DP IB Maths: AI HL

3.6 Matrix Transformations

Contents

3.6.1 Matrix Transformations

Transformation by a Matrix

What is a transformation matrix?

- A transformation matrix is used to determine the coordinates of an **image** from the **transformation** of an **object**
  - Commonly used transformation matrices include
    - reflections, rotations, enlargements and stretches
- (In 2D) a multiplication by any 2×2 matrix could be considered a transformation (in the 2D plane)
- An individual point in the plane can be represented as a position vector, $\begin{pmatrix} x \\ y \end{pmatrix}$
  - Several points, that create a shape say, can be written as a position matrix $\begin{pmatrix} x_1 & x_2 & x_3 & \dots \\ y_1 & y_2 & y_3 & \dots \end{pmatrix}$
- A matrix transformation will be of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$
  - where $\begin{pmatrix} x \\ y \end{pmatrix}$ represents any point in the 2D plane
  - $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} e \\ f \end{pmatrix}$ are given matrices

How do I find the coordinates of an image under a transformation?

- The coordinates (x', y') - the image of the point (x, y) under the transformation with matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} e \\ f \end{pmatrix}$ - are given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

- Similarly, for a position matrix

$$\begin{pmatrix} x'_1 & x'_2 & x'_3 & \dots \\ y'_1 & y'_2 & y'_3 & \dots \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 & x_2 & x_3 & \dots \\ x_1 & x_2 & x_3 & \dots \end{pmatrix} + \begin{pmatrix} e & e & e & \dots \\ f & f & f & \dots \end{pmatrix}$$

- If you use this method then remember to add e and *f* to each column
- A GDC can be used for matrix multiplication
  - If matrices involved are small, it may be as quick to do this manually

- **STEP 1**
  Determine the transformation matrix (**T**) and the position matrix (**P**)
  The transformation matrix, if uncommon, will be given in the question
  The position matrix is determined from the coordinates involved, it is best to have the coordinates in order, to avoid confusion

- **STEP 2**
  Set up and perform the matrix multiplication and addition required to determine the image position matrix, **P'**
  P' = **TP**

- **STEP 3**
  Determine the coordinates of the image from the image position matrix, **P'**

How do I find the coordinates of the original point given the image under a transformation?

- To 'reverse' a transformation we would need the **inverse transformation matrix**
  - i.e. $\mathbf{T}^{-1}$
  - For a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the inverse is given by $\frac{1}{\det \boldsymbol{T}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
    - where $\det \boldsymbol{T} = ad - bc$
  - A GDC can be used to work out inverse matrices
- You would rearrange $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$
  - $\frac{1}{\det \boldsymbol{T}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\left[\begin{pmatrix} x' \\ y' \end{pmatrix} - \begin{pmatrix} e \\ f \end{pmatrix}\right] = \begin{pmatrix} x \\ y \end{pmatrix}$

Worked example

A quadrilateral, Q, has the four vertices A(2, 5), B(5, 9), C(11, 9) and D(8, 5).

Find the coordinates of the image of Q under the transformation $\boldsymbol{T}=\begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$.

STEP 1: Determine the transformation and position matrices

$$\underset{\sim}{T} = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \qquad \underset{\sim}{P} = \begin{array}{c} \begin{pmatrix} 2 & 5 & 11 & 8 \\ 5 & 9 & 9 & 5 \end{pmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ A & B & C & D \end{array} \end{array}$$

STEP 2: $\underset{\sim}{P}' = \underset{\sim}{T}\underset{\sim}{P}$

$$\underset{\sim}{P}' = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 & 11 & 8 \\ 5 & 9 & 9 & 5 \end{pmatrix}$$

$$\underset{\sim}{P}' = \begin{pmatrix} 6-5 & 15-9 & 33-9 & 24-5 \\ -2+10 & -5+18 & -11+18 & -8+10 \end{pmatrix}$$

$$\underset{\sim}{P}' = \begin{pmatrix} 1 & 6 & 24 & 19 \\ 8 & 13 & 7 & 2 \end{pmatrix}$$

Alternatively use a GDC for matrix multiplication

STEP 3: Determine the image coordinates from P'

$$\boxed{A'(1,8) \quad B'(6,13) \quad C'(24,7) \quad D'(19,2)}$$

Matrices of Geometric Transformations

What is meant by a geometric transformation?

- The following transformations can be represented (in 2D) using **multiplication** of a **2×2** matrix
  - rotations (about the origin)
  - reflections
  - enlargements
  - (horizontal) stretches parallel to the x-axis
  - (vertical) stretches parallel to the y-axis
- The following transformations can be represented (in 2D) using **addition** of a **2×1** matrix
  - translations

What are the matrices for geometric transformations?

- All of the following transformation matrices are given in the **formula booklet**
- **Rotation**
  - Anticlockwise (or counter-clockwise) through angle θ about the origin
    - $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
  - Clockwise through angle θ about the origin
    - $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$
  - In both cases
    - $\theta > 0$
    - θ may be measured in degrees or radians
- **Reflection**
  - In the line $y = (\tan\theta)x$
    - $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$
  - θ may be measured in degrees or radians
  - for a reflection in the x-axis, θ = 0° (0 radians)
  - for a reflection in the y-axis, θ = 90° (π/2 radians)
- **Enlargement**
  - Scale factor *k*, centre of enlargement at the origin (0, 0)
    - $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$
- **Horizontal stretch** (or stretch parallel to the x-axis)
  - Scale factor *k*

- $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$

- **Vertical stretch** (or stretch parallel to the y-axis)
  - Scale factor *k*
    - $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
- **Translation (vector)**
  - *p* units in the (positive) x-direction
  - *q* units in the (positive) y direction
    - $\begin{pmatrix} p \\ q \end{pmatrix}$
    - This is not given in the formula booklet

How do I solve problems involving geometric transformations?

- The matrix equations involved in problems will be of the form
  - **P'=AP** or
  - **P'=AP+b** where **b** is a translation vector
    - (sometimes called an **affine** transformation)
  - where
    - **P** is the position vector of the object coordinates
    - **P'** is the position vector of the image coordinates
    - **A** is the transformation matrix
    - **b** is a translation vector
- Problems may ask you to
  - find the coordinates of point(s) on the image
  - find the coordinates of point(s) on the object using an inverse matrix ($\mathbf{A}^{-1}$)
  - deduce/identify a matrix corresponding to one of the common geometric transformations
    - E.g. Find the matrix of a rotation of 45° clockwise about the origin

Worked example

Triangle PQR has coordinates P(-1, 4), Q(5, 4) and R(2, -1).

The transformation ***T*** is a reflection in the line $y = x\sqrt{3}$.

a) Find the matrix ***T*** that represents a reflection in the line $y = x\sqrt{3}$.

From formula booklet: Reflection in line $y = (\tan\theta)x$

is $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

$y = x\sqrt{3}$, $\therefore \tan\theta = \sqrt{3}$, $\theta = 60°$

$\therefore \underset{\sim}{T} = \begin{pmatrix} \cos 120° & \sin 120° \\ \sin 120° & -\cos 120° \end{pmatrix}$

$$\boxed{\underset{\sim}{T} = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}}$$

b) Find the position matrix, ***P'***, representing the coordinates of the images of points P, Q and R under the transformation ***T.***

$\underset{\sim}{P}' = \underset{\sim}{T}\underset{\sim}{P}$ ("$\underset{\sim}{P}' = \underset{\sim}{A}\underset{\sim}{P}$")

$$\therefore \underset{\sim}{P}' = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \begin{array}{c} \begin{pmatrix} -1 & 5 & 2 \\ 4 & 4 & -1 \end{pmatrix} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ P & Q & R \end{array} \end{array} \leftarrow \text{position matrix P}$$

(Use a GDC for matrix multiplication)

$$\boxed{\underset{\sim}{P}' = \begin{pmatrix} \frac{1}{2}(1+4\sqrt{3}) & -\frac{1}{2}(5-4\sqrt{3}) & -\frac{1}{2}(2+\sqrt{3}) \\ \frac{1}{2}(4-\sqrt{3}) & \frac{1}{2}(4+5\sqrt{3}) & -\frac{1}{2}(1-2\sqrt{3}) \end{pmatrix}}$$

Be careful copying a calculator display

$$-\frac{2+\sqrt{3}}{2} \neq \frac{-2+\sqrt{3}}{2}$$

Matrices of Composite Transformations

The order in which transformations occur can lead to different results – for example a reflection in the x-axis followed by clockwise rotation of 90° is different to rotation first, followed by the reflection.

Therefore, when one transformation is followed by another order is critical.

What is a composite transformation?

- A composite function is the result of applying more than one function to a point or set of points
  - e.g. a **rotation**, followed by an **enlargement**
- It is possible to find a **single** composite function **matrix** that does the same job as applying the individual transformation matrices

How do I find a single matrix representing a composite transformation?

- Multiplication of the transformation matrices
- However, the order in which the matrices is important
  - If the transformation represented by matrix **M** is applied first, and is then followed by another transformation represented by matrix **N**
    - the composite matrix is **NM**
      e. **P' = NMP**
      (**NM** is not necessarily equal to **MN**)
    - The matrices are **applied** right to left
    - The composite function matrix is **calculated** left to right
  - Another way to remember this is, starting from **P**, always **pre-multiply** by a transformation matrix
    - This is the same as applying **composite functions** to a value
    - The function (or matrix) furthest to the right is applied first

How do I apply the same transformation matrix more than once?

- If a transformation, represented by the matrix **T**, is applied twice we would write the composite transformation matrix as $\mathbf{T}^2$
  - $\mathbf{T}^2 = \mathbf{TT}$
- This would be the case for any number of repeated applications
  - $\mathbf{T}^5$ would be the matrix for five applications of a transformation
- A GDC can quickly calculate $\mathbf{T}^2$, $\mathbf{T}^5$, etc
- Problems may involve considering patterns and sequences formed by repeated applications of a transformation
  - The coordinates of point(s) follow a particular pattern
    - (20, 16) – (10, 8) – (5, 4) – (2.5, 2) ...
  - The area of a shape increases/decreases by a constant factor with each application

e.g. if one transformation doubles the area then three applications will increase the (original) area eight times (2^3)

Worked example

The matrix **E** represents an enlargement with scale factor 0.25, centred on the origin.
The matrix **R** represents a rotation, 90° anticlockwise about the origin.

a) Find the matrix, **C**, that represents a rotation, 90° anticlockwise about the origin followed by an enlargement of scale factor 0.25, centred on the origin.

$$\underset{\sim}{C} = \underset{\sim}{E}\,\underset{\sim}{R}$$

$$\underset{\sim}{C} = \underbrace{\begin{pmatrix} 0.25 & 0 \\ 0 & 0.25 \end{pmatrix}}_{\substack{\text{enlargement} \\ k=0.25}} \underbrace{\begin{pmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{pmatrix}}_{\substack{\text{rotation, anti-clockwise} \\ \theta=90°}}$$

Use the formula booklet

$$\underset{\sim}{C} = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.25 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Use a GDC for matrix multiplication

$$\boxed{\therefore \underset{\sim}{C} = \begin{pmatrix} 0 & -0.25 \\ 0.25 & 0 \end{pmatrix}}$$

b) A square has position matrix $T_0 = \begin{pmatrix} 0 & 0 & 256 & 256 \\ 0 & 256 & 256 & 0 \end{pmatrix}$. T_n represents the position matrix of the image square after it has been transformed n times by matrix **C**. Find T_4

$$\underset{\sim}{T_4} = \underset{\sim}{C}^4 \underset{\sim}{T_0} = \begin{pmatrix} 0 & -0.25 \\ 0.25 & 0 \end{pmatrix}^4 \begin{pmatrix} 0 & 0 & 256 & 256 \\ 0 & 256 & 256 & 0 \end{pmatrix}$$

Use a GDC, typing this in carefully as one calculation

$$\boxed{\therefore \underset{\sim}{T_4} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}}$$

c) Find the single transformation matrix that would map T_4 to T_0.

$\underset{\sim}{T}_4$ to $\underset{\sim}{T}_0$ would be the inverse of $\underset{\sim}{C}^4$.

(Note that $[C^4]^{-1}$ does not mean $\underset{\sim}{C}^{-4}$)

Use a GDC to find $[C^4]^{-1}$ in one calculation

$$\boxed{[C^4]^{-1} = \begin{pmatrix} 256 & 0 \\ 0 & 256 \end{pmatrix}}$$

3.6.2 Determinant of a Transformation Matrix

Determinant of a Transformation Matrix

What is a determinant?

- For the 2×2 matrix $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$
    - the determinant is det $\boldsymbol{A}$ = $ad - bc$

What does the determinant of a transformation matrix ($\boldsymbol{A}$) represent?

- The **absolute value** of the **determinant** of a transformation matrix is the **area scale factor**
    - Area scale factor = $|\det \boldsymbol{A}|$
- The area of the **image** will be **product** of the **area** of the **object** and $|\det \boldsymbol{A}|$
    - Area of image = $|\det \boldsymbol{A}|$ × Area of object
- Note the area will reduce if $|\det \boldsymbol{A}| < 1$
- If the determinant is **negative** then the **orientation** of the shape will be **reversed**
    - For example: the shape has been reflected

How do I solve problems involving the determinant of a transformation matrix?

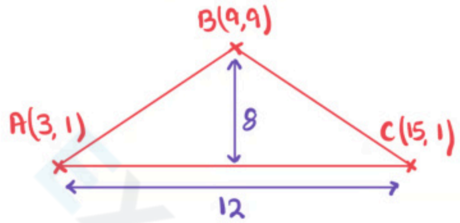
- Problems may involve comparing areas of **objects** and **images**
    - This could be as a percentage, proportion, etc
- Missing value(s) from the transformation matrix (and elsewhere) can be deduced if the determinant of the transformation matrix is known
- Remember to use the **absolute value** of the determinant
    - This can lead to multiple answers to equations
    - Use your GDC to solve these

Worked example

An isosceles triangle has vertices A(3, 1), B(15, 1) and C(9, 9).

a) Find the area of the isosceles triangle.

A sketch or plot on GDC will help find the area



Area $= \frac{1}{2} \times 12 \times 8$ ("$A = \frac{1}{2}bh$")

$\boxed{\therefore \text{ Area of } \Delta ABC = 48 \text{ square units}}$

b) Triangle $\triangle$ABC is transformed using the matrix $\boldsymbol{T} = \begin{pmatrix} 3 & 2 \\ -1 & 2 \end{pmatrix}$. Find the area of the transformed triangle.

Area scale factor is $|\det \underset{\sim}{T}|$

$|\det \underset{\sim}{T}| = 3 \times 2 - 2 \times -1 = 8$

$\therefore$ Area of image $= 48 \times 8 = 384$

$\boxed{\text{Area of transformed triangle} = 384 \text{ square units}}$

c) Triangle $\triangle$ABC is now transformed using the matrix $\boldsymbol{U} = \begin{pmatrix} a & -2 \\ 3 & a^2 \end{pmatrix}$ where $a \in \mathbb{Z}$. Given that the area of the image is twice as large as the area of the object, find the value of a.

$\det \underset{\sim}{U} = a \times a^2 - -2 \times 3 = a^3 + 6$

$\therefore |a^3+6| = 2$

For $a^3+6=2$, $a^3=-4$, $a \notin \mathbb{Z}^-$, reject

For $a^3+6=-2$, $a^3=-8$, $a=-2$, $a \in \mathbb{Z}^-$

$\boxed{\therefore a=-2}$