DP IB Maths: AA HL

3.5 Trigonometric Functions & Graphs

Contents

3.5.1 Graphs of Trigonometric Functions

Graphs of Trigonometric Functions

What are the graphs of trigonometric functions?

- The trigonometric functions sin, cos and tan all have special **periodic graphs**
- You'll need to know their properties and how to sketch them for a given domain in either **degrees** or **radians**
- Sketching the trigonometric graphs can help to
  - Solve trigonometric equations and find all solutions
  - Understand transformations of trigonometric functions

What are the properties of the graphs of sin x and cos x?

- The graphs of sin x and cos x are both **periodic**
  - They **repeat every 360°** (2π radians)
  - The angle will always be on the x-axis
    - Either in degrees or radians
- The graphs of sin x and cos x are always in the **range −1 ≤ *y* ≤ 1**
  - **Domain:** $\{x \mid x \in \mathbb{R}\}$
  - **Range:** $\{y \mid -1 \leq y \leq 1\}$
  - The graphs of sin x and cos x are identical however one is a **translation** of the other
    - sin x passes through the origin
    - cos x passes through (0, 1)
- The **amplitude** of the graphs of sin x and cos x is 1

What are the properties of the graph of tan x?

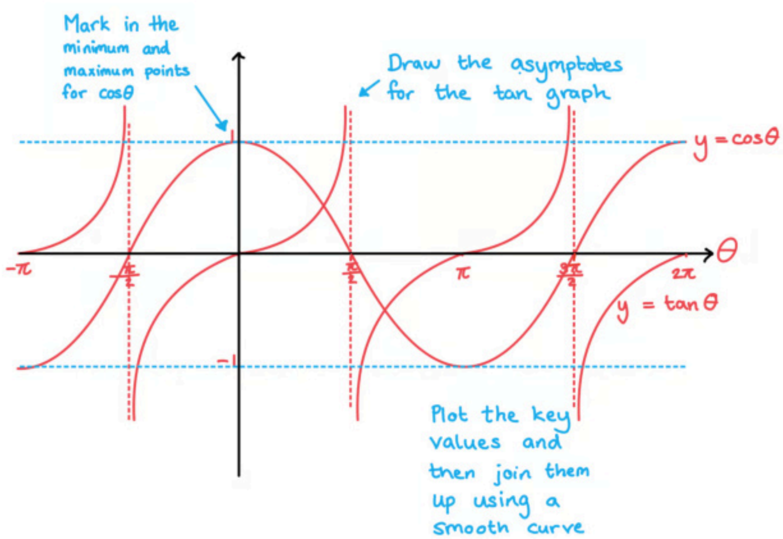
- The graph of tan x is **periodic**
  - It **repeats every 180°** (π radians)
  - The angle will always be on the x-axis
    - Either in degrees or radians
- The graph of tan x is **undefined** at the points ± 90°, ± 270° etc
  - There are **asymptotes** at these points on the graph
  - In radians this is at the points $\pm\frac{\pi}{2}$, $\pm\frac{3\pi}{2}$ etc
- The range of the graph of tan x is
  - **Domain:** $\left\{x \mid x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\right\}$
  - **Range:** $\{y \mid y \in \mathbb{R}\}$

How do I sketch trigonometric graphs?

- You may need to sketch a trigonometric graph so you will need to remember the key features of each one
- The following steps may help you sketch a trigonometric graph
  - STEP 1: Check whether you should be working in degrees or radians
    - You should check the domain given for this
    - If you see π in the given domain then you should work in radians
  - STEP 2: Label the x-axis in multiples of 90°
    - This will be multiples of $\frac{\pi}{2}$ if you are working in radians
    - Make sure you cover the whole domain on the x-axis
  - STEP 3: Label the y-axis
    - The range for the y-axis will be $-1 \le y \le 1$ for sin or cos
    - For tan you will not need any specific points on the y-axis
  - STEP 4: Draw the graph
    - Knowing exact values will help with this, such as remembering that sin(0) = 0 and cos(0) = 1
    - Mark the important points on the axis first
    - If you are drawing the graph of tan x put the asymptotes in first
    - If you are drawing sin x or cos x mark in where the maximum and minimum points will be
    - Try to keep the symmetry and rotational symmetry as you sketch, as this will help when using the graph to find solutions

Worked example

Sketch the graphs of y = cosθ and y = tanθ on the same set of axes in the interval -π ≤ θ ≤ 2π. Clearly mark the key features of both graphs.



Using Trigonometric Graphs

How can I use a trigonometric graph to find extra solutions?

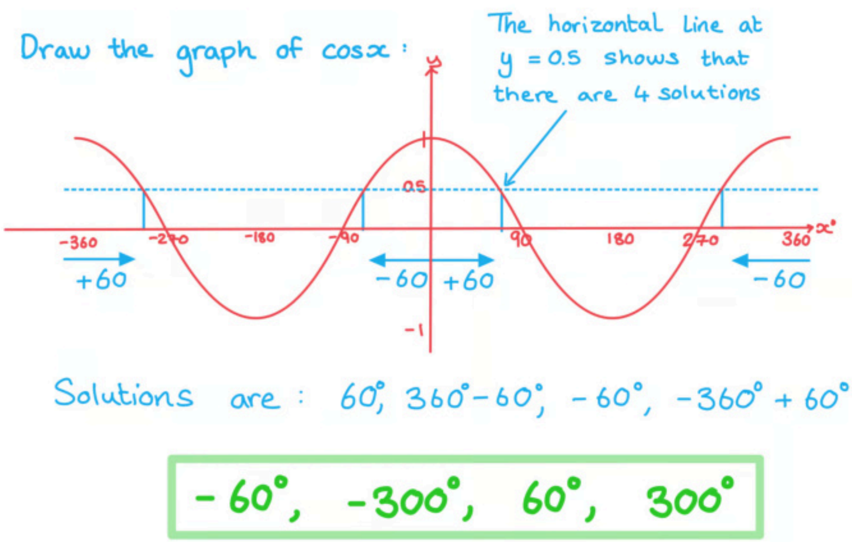
- Your calculator will only give you the first solution to a problem such as $\sin^{-1}(0.5)$
  - This solution is called the **primary value**
- However, due to the **periodic** nature of the trig functions there could be an infinite number of solutions
  - Further solutions are called the **secondary values**
- This is why you will be given a **domain** (interval) in which your solutions should be found
  - This could either be in degrees or in radians
    - If you see π or some multiple of π then you must work in radians
- The following steps will help you use the **trigonometric graphs** to find **secondary values**
  - STEP 1: Sketch the graph for the given function and interval
    - Check whether you should be working in degrees or radians and label the axes with the key values
  - STEP 2: Draw a horizontal line going through the y-axis at the point you are trying to find the values for
    - For example if you are looking for the solutions to $\sin^{-1}(-0.5)$ then draw the horizontal line going through the y-axis at -0.5
    - The number of times this line cuts the graph is the number of solutions within the given interval
  - STEP 3: Find the primary value and mark it on the graph
    - This will either be an exact value and you should know it
    - Or you will be able to use your calculator to find it
  - STEP 4: Use the symmetry of the graph to find all the solutions in the interval by adding or subtracting from the key values on the graph

What patterns can be seen from the graphs of trigonometric functions?

- The graph of sin x has rotational symmetry about the origin
  - So $\sin(-x) = -\sin(x)$
  - $\sin(x) = \sin(180° - x)$ or $\sin(\pi - x)$
- The graph of cos x has reflectional symmetry about the y-axis
  - So $\cos(-x) = \cos(x)$
  - $\cos(x) = \cos(360° - x)$ or $\cos(2\pi - x)$
- The graph of tan x repeats every 180° (π radians)
  - So $\tan(x) = \tan(x \pm 180°)$ or $\tan(x \pm \pi)$
- The graphs of sin x and cos x repeat every 360° (2π radians)
  - So $\sin(x) = \sin(x \pm 360°)$ or $\sin(x \pm 2\pi)$
  - $\cos(x) = \cos(x \pm 360°)$ or $\cos(x \pm 2\pi)$

Worked example

One solution to cos x = 0.5 is 60°. Find all the other solutions in the range -360° ≤ x ≤ 360°.



3.5.2 Transformations of Trigonometric Functions

Transformations of Trigonometric Functions

What transformations of trigonometric functions do I need to know?

- As with other graphs of functions, trigonometric graphs can be transformed through **translations, stretches** and **reflections**
- **Translations** can be either horizontal (parallel to the x-axis) or vertical (parallel to the y-axis)
  - For the function **y = sin (x)**
    - A **vertical** translation of ***a* units** in the **positive direction** (up) is denoted by **y = sin (x) + *a***
    - A **vertical** translation of ***a* units** in the **negative direction** (down) is denoted by **y = sin (x) – *a***
    - A **horizontal** translation in the **positive direction** (right) is denoted by **y = sin (x – *a*)**
    - A **horizontal** translation in the **negative direction** (left) is denoted by **y = sin (x + *a*)**
- **Stretches** can be either horizontal (parallel to the x-axis) or vertical (parallel to the y-axis)
  - For the function **y = sin (x)**
    - A **vertical** stretch of a **factor *a* units** is denoted by **y = *a* sin (x)**
    - A **horizontal** stretch of a **factor *a* units** is denoted by **y = sin** $\left(\frac{x}{a}\right)$
- **Reflections** can be either across the x-axis or across the y-axis
  - For the function **y = sin (x)**
    - A **reflection** across the x-axis is denoted by **y = – sin (x)**
    - A **reflection** across the y-axis is denoted by **y = sin (–x)**

What combined transformations are there?

- **Stretches** in the horizontal and vertical direction are often combined
- The functions ***a* sin(*bx*)** and ***a* cos(*bx*)** have the following properties:
  - The **amplitude** of the graph is $|a|$
  - The **period** of the graph is $\frac{360}{b}°$ (or $\frac{2\pi}{b}$ rad)
- **Translations** in both directions could also be combined with the stretches
- The functions ***a* sin(*b*(*x* – c)) + *d*** and ***a* cos(*b*(*x* – c)) + *d*** have the following properties:
  - The **amplitude** of the graph is $|a|$
  - The **period** of the graph is $\frac{360}{b}°$ (or $\frac{2\pi}{b}$)
  - The translation in the horizontal direction is c
  - The translation in the vertical direction is *d*
    - *d* represents the **principal axis** (the line that the function fluctuates about)
- The function ***a* tan(*b*(*x* – c)) + *d*** has the following properties:
  - The **amplitude** of the graph does not exist

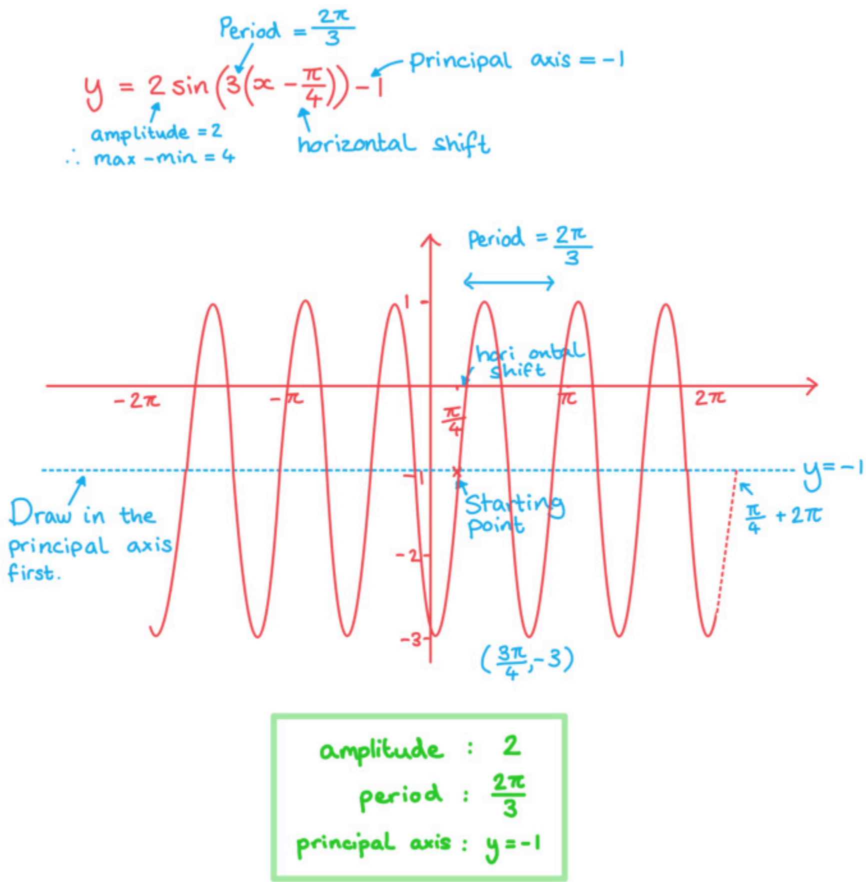
- The **period** of the graph is $\frac{180}{b}$° (or $\frac{2\pi}{b}$)
- The translation in the horizontal direction is *c*
- The translation in the vertical direction (principal axis) is *d*

How do I sketch transformations of trigonometric functions?

- Sketch the graph of the original function first
- Carry out each transformation separately
  - The **order** in which you carry out the transformations is important
  - Given the form ***y* = *a* sin(*b*(*x* – c)) + *d*** carry out any **stretches** first, **translations** next and **reflections** last
    - If the function is written in the form ***y* = *a* sin(*bx* – *bc*) + *d*** factorise out the coefficient of *x* before carrying out any transformations
  - Use a very light pencil to mark where the graph has moved for each transformation
- It is a good idea to mark in the principal axis the lines corresponding to the maximum and minimum points first
  - The **principal axis** will be the line ***y* = *d***
  - The **maximum points** will be on the line ***y* = *d* + *a***
  - The **minimum points** will be on the line ***y* = *d* – *a***
- Sketch in the new transformed graph
- Check it is correct by looking at some key points from the **exact values**

Worked example

Sketch the graph of $y = 2\sin\left(3\left(x - \frac{\pi}{4}\right)\right) - 1$ for the interval $-2\pi \le x \le 2\pi$. State the amplitude, period and principal axis of the function.



3.5.3 Modelling with Trigonometric Functions

Modelling with Trigonometric Functions

What can be modelled with trigonometric functions?

- Anything that oscillates (fluctuates periodically) can be modelled using a trigonometric function
  - Normally some transformation of the sine or cosine function
- Examples include:
  - $D(t)$ is the depth of water at a shore *t* hours after midnight
  - $T(d)$ is the temperature of a city *d* days after the 1st January
  - $H(t)$ is vertical height above ground of a person *t* seconds after entering a Ferris wheel
- Notice that the x-axis will not always contain an angle
  - In the examples above time or number of days would be on the x-axis
  - Depth of the water, temperature or vertical height would be on the y-axis

What are the parameters of trigonometric models?

- A **trigonometric model** could be of the form
  - $f(x) = a\sin(b(x - c)) + d$
  - $f(x) = a\cos(b(x - c)) + d$
  - $f(x) = a\tan(b(x - c)) + d$
- The *a* represents the **amplitude** of the function
  - The bigger the value of *a* the bigger the **range** of values of the function
  - For the function $a\tan(b(x - c)) + d$ the amplitude is undefined
- The *b* determines the **period** of the function
  - Period $= \dfrac{360°}{b} = \dfrac{2\pi}{b}$
  - The bigger the value of *b* the quicker the function repeats a cycle
- The *c* represents the **horizontal shift**
- The *d* represents the **vertical shift**
  - This is the **principal axis**

What are possible limitations of a trigonometric model?

- The amplitude is the same for each cycle
  - In real-life this might not be the case
  - The function might get closer to the value of *d* over time
- The period is the same for each cycle
  - In real-life this might not be the case
  - The time to complete a cycle might change over time

Worked example

The water depth, *D*, in metres, at a port can be modelled by the function

$$D(t) = 3\sin\left(15°(t-2)\right) + 12, \qquad 0 \le t < 24$$

where *t* is the elapsed time, in hours, since midnight.

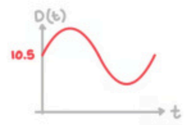
a) Write down the depth of the water at midnight.

Substitute $t = 0$ for midnight:

$D(0) = 3\sin(15(0-2)) + 12$

$= 3\sin(-30) + 12$

$= 3\left(-\frac{1}{2}\right) + 12 \quad = 10.5$



$D = 10.5\text{ m}$

b) Find the minimum water depth and the number of hours after midnight that this depth occurs.

$D(t) = 3\sin(15(t-2)) + 12$

3 ↑ amplitude; 12 ↑ principal axis

Principal axis is at $y = 12$

amplitude is 3   Minimum $= 12 - 3 = 9$

Let $D(t) = 9$

$$3\sin(15(t-2)) + 12 = 9$$

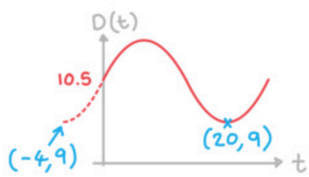
$$3\sin(15(t-2)) = -3$$

$$\sin(15(t-2)) = -1$$

$$15(t-2) = -90$$

$$t = -4 + 24n$$

24n ↑ cycle repeats every 24 hours



Minimum = 9m
20 hours after midnight

c) Calculate how long the water depth is at least 13.5 m each day.

Let $D(t) = 13.5$

$$3\sin(15(t-2)) + 12 = 13.5$$

$$3\sin(15(t-2)) = 1.5$$

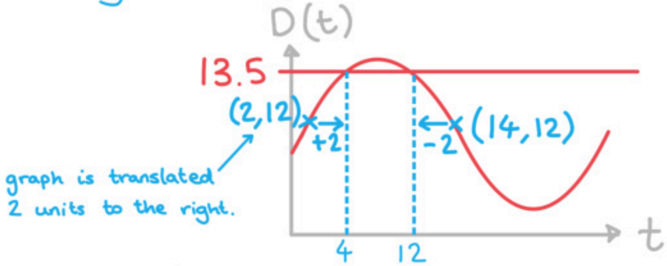
$$\sin(15(t-2)) = 0.5$$

$$15(t-2) = 30$$

$$t = 4 + 24n$$

cycle repeats every 24 hours

Use symmetry and properties of the graph to find secondary value of t:



$t = 4$ and $t = 12$

Find the difference between the times.

$12 - 4 = 8$

8 hours