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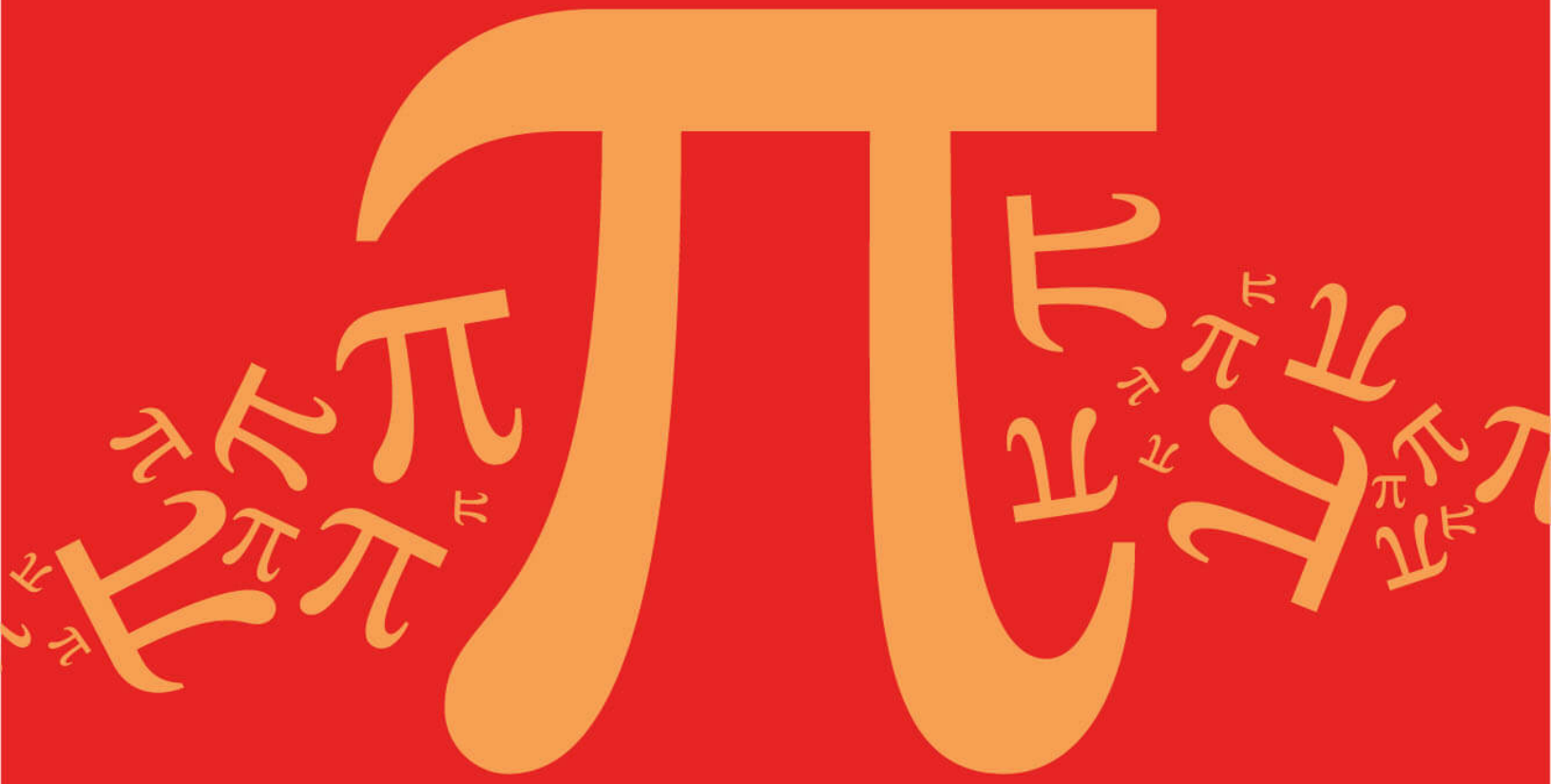
3.5 Voronoi Diagrams



IB Maths - Revision Notes

3.5.1 Voronoi Diagrams

Drawing Voronoi Diagrams

What are Voronoi Diagrams?

- A **Voronoi diagram** shows the region containing the set of all points which are **closer** to one given **site** than to any other **site** on the diagram
  - A **site** is located at the coordinates of a specific place of interest on a Voronoi diagram
- It will be partitioned into a number of **regions**
  - These regions are often called **Voronoi cells** and will be **polygons**
    - There will be the same number of **regions** as **sites** on the diagram
  - For example, if a city contains five parks a Voronoi diagram could be drawn for that city dividing it into five regions based on their closest park
- The **edges** of each region will be the **perpendicular bisector** of two of the sites
  - The **edges** may also be called **boundaries**
- The **vertices** of each region are the **intersections** of **three** of these perpendicular bisectors
  - The perpendicular bisectors of three individual points will always intersect at the point that is **equidistant** from the three points

How are Voronoi diagrams drawn for three sites?

- You will **not** be expected to draw a Voronoi diagram from scratch, however you should understand how one is constructed
  - First, the perpendicular bisector of the line segment joining each pair of sites will be constructed
    - These should be constructed using dashed lines as only a part of each line will be needed for the final diagram
  - The **points of intersection** of these perpendicular bisectors will create the **vertices**
  - Each perpendicular bisector should stop when it meets another perpendicular bisector
    - Remove the part of the perpendicular bisector that is not in the region of the two sites
    - No perpendicular bisector should cross over another
    - This will form the **regions**, or **cells**

How are Voronoi diagrams drawn for more than three sites?

- It is challenging to draw a Voronoi diagram from scratch if it has **more than three sites**
- In this case it is easiest to draw the Voronoi diagram for three sites first and then add the next sites one by one following these steps
  - STEP 1: The fourth site will be in one of the cells containing an existing site
    - Draw the perpendicular bisector of the line segment between these two sites
  - STEP 2: Stop this new line at the point where it meets an existing **boundary** in the Voronoi diagram

- STEP 3: There will now be an existing edge in the region of the new site
  - This should be **shortened** to meet the new boundary
- STEP 4: The fourth site will now be in the same cell as a different existing site
  - Draw the perpendicular bisector of the line segment between these two sites
  - This is the step you will most likely carry out in an exam

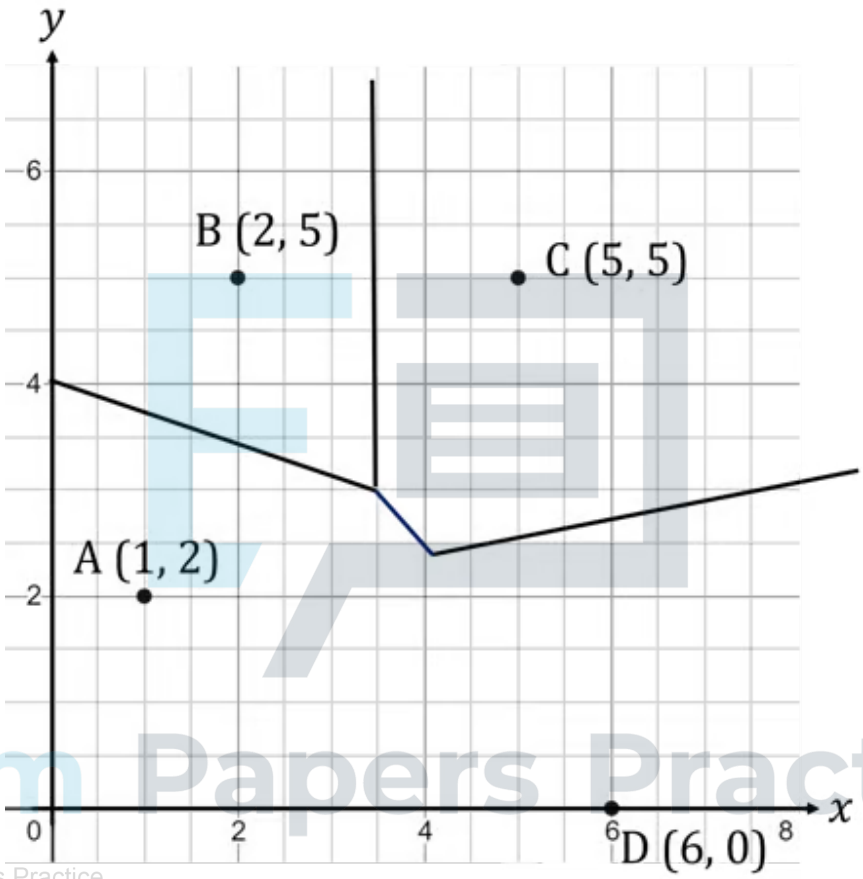
- You may be asked to find the **equation of a missing edge**
  - This will mean finding the equation of the **perpendicular bisector** between the **two sites** that are both within **one region**
- You may be asked to add the **location of a missing site** to the Voronoi diagram
  - This will mean using the given **edge** of one or two of the regions and finding the second site that would make this edge a perpendicular bisector
    - Draw a perpendicular line from the site to the edge
    - Check the distance of this line and then continue it on the other site of the edge for the same distance
    - This will be the location of your new site
  - You may need to find the gradients of the edges you have and then use the negative reciprocal to find the gradient of the perpendicular bisector of the current and new site

Exam Tip

- Make sure that you have a straight edge and an eraser with you in the exam so that any perpendicular bisectors that you draw are clear and any mistakes that are made can be erased
- If you are asked to adjust a given Voronoi diagram and a perpendicular bisector that needs to be removed or shortened, you can put a series of little lines along it to indicate that it is crossed out

Worked example

The Voronoi diagram below shows sites A, B, C and D.



a) Explain how you know that the Voronoi diagram is incomplete.

The Voronoi diagram has four sites but only three Voronoi cells.

b) Find the equation of the line which would complete the Voronoi cell containing site A.
Give your answer in the form $ax + by + d = 0$ where $a, b, d \in \mathbb{Z}$.

Sites A and D are both in the same region so find the perpendicular bisector of the line segment connecting A and D.

A: (1, 2) D: (6, 0)

Find the midpoint:

$$MP = \left(\frac{1+6}{2}, \frac{2+0}{2}\right) = (3.5, 1)$$

gradient AD

$$m_{AD} = \frac{0-2}{6-1} = -\frac{2}{5} \quad \therefore \quad m_{\perp AD} = \frac{5}{2}$$

Perpendicular gradient.

Sub MP and $m_{\perp AD}$ into equation for a straight line:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = \frac{5}{2}\left(x - \frac{7}{2}\right)$$

multiply by 2 to remove the fraction and rearrange

$$2y - 2 = 5x - \frac{35}{2}$$

$$4y - 4 = 10x - 35$$

$$10x - 4y - 31 = 0$$

Interpreting Voronoi Diagrams

What is a Voronoi diagram used for?

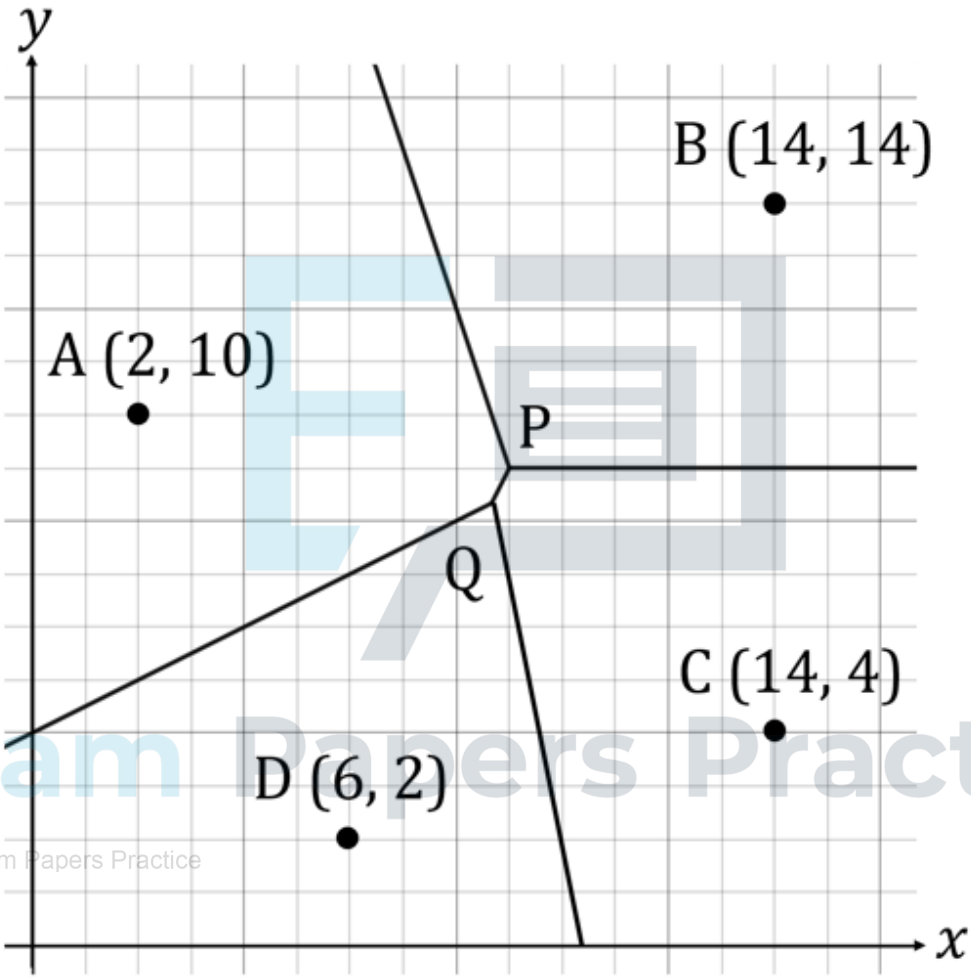
- Voronoi diagrams are often used in land management to work out where the best location would be according to where sites are already situated
- They can show where to put something to make sure that it is
  - Closest to a particular site
  - Closer to one site than another
  - Equidistant from two or three specific sites
  - As far as possible from any other site

What do I need to know about Voronoi diagrams?

- You may be asked to find the shortest distance from a point to its closest site
  - Use Pythagoras' Theorem to find the distance between the given coordinate and the site in the same region as it
  - If the coordinate is on an edge then there will be two sites **equidistant** from it
- You may be asked to find the point which is furthest from any of the sites
  - This will be one of the vertices
  - To choose which vertex look at which is the centre of the **largest empty circle**
- You may be asked to estimate the success of a new site
  - This is done by looking at the data for the **nearest site**
  - The prediction for the new site would be assumed to be the same
    - This is called **nearest neighbour interpolation**

Worked example

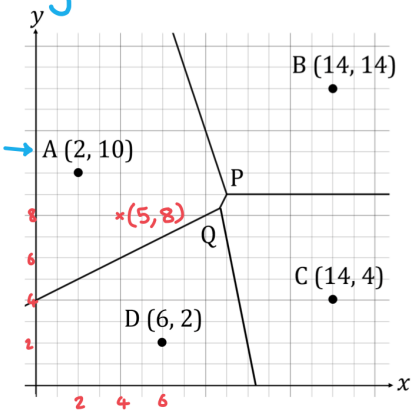
The Voronoi diagram below shows the four sites A, B, C and D with coordinates (2, 10), (14, 14), (14, 4), and (6, 2) respectively. 1 unit represents 10 km.



i) State which site a new business opening at the coordinate (5, 8) should look at to predict future sales.

Plot the point and look for the site in the same region:

The new business is in the same region as site A.



Site A

ii) Find the shortest distance from the point (5, 8) to its nearest site.

The point (5, 8) is closest to site A.

Formula for distance between two points:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

P: (5, 8) A: (2, 10)

x_1 y_1 x_2 y_2

Sub coordinates:

$$d = \sqrt{(2-5)^2 + (10-8)^2}$$

$$= \sqrt{(-3)^2 + (2)^2} = \sqrt{13}$$

distance $= \sqrt{13} \times 10\text{km} = 36.055\ldots\text{km}$

distance = 36.1 km (3 s.f.)

3.5.2 Toxic Waste Dump Problem

Toxic Waste Dump Problem

What is the toxic waste dump problem?

- The **toxic waste dump problem** is the name given to the general idea of finding the point on a **Voronoi diagram** which is furthest from any of the **sites**
  - A **site** is the coordinates of a specific place of interest on a Voronoi diagram
- It is given this name because of the common problem of finding a place to put a toxic waste dump that is **equally far** away from any inhabited area
  - For example, if a province contains five towns a Voronoi diagram could be used to find the point within the province which is furthest from each town
- The toxic waste dump problem is more of an idea than a specific problem
  - The same concept could be applied to other contexts such as
    - Finding a position for a new supermarket that is equally far from all competitors
    - Finding a place to plant a new tree that is equally far from other trees competing for water resources
    - Finding the quietest place to enjoy a picnic that is equally far from other noisy groups of people
  - Note that the term **equally far** is used in all of the above examples

How is a Voronoi diagram used to find the furthest point from any site?

- Within any Voronoi diagram the furthest point from any site will always be either
  - one of the cell vertices, or
  - somewhere on a boundary of the diagram
- In an IB exam, the solution will always be one of the **cell vertices**
- To find the furthest point you will need to consider each of the cell vertices separately and find which one is furthest from all of the sites
- This is done by constructing the **largest empty circle**

What is the largest empty circle?

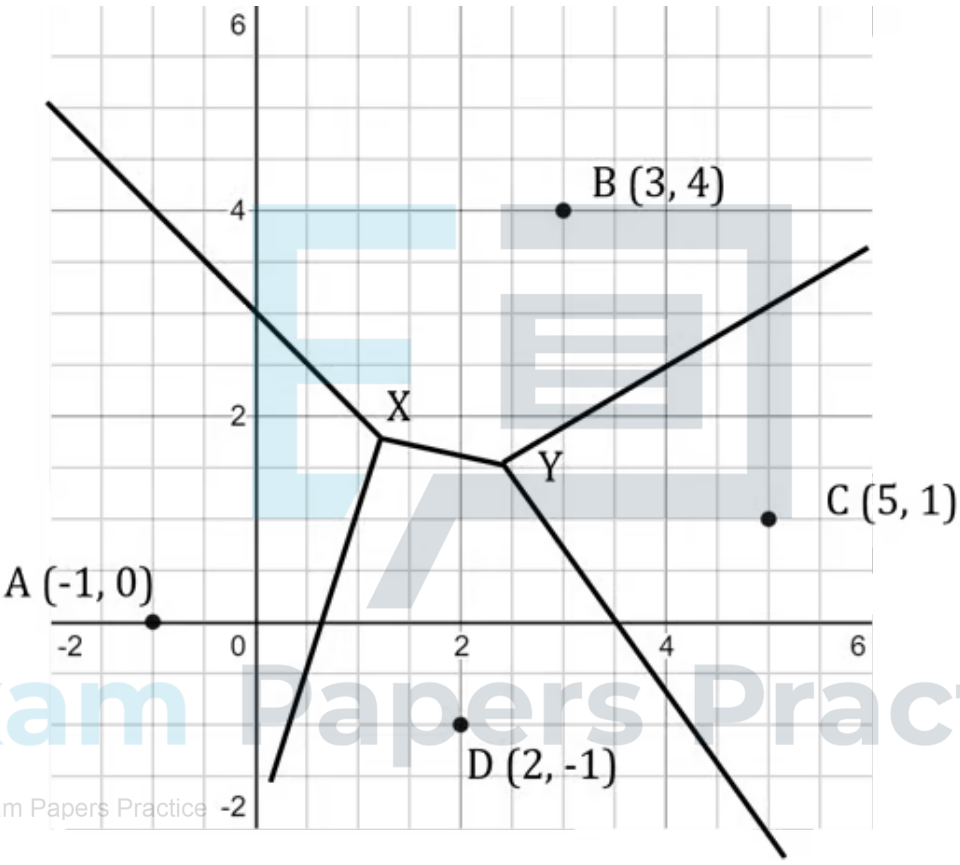
- The **largest empty circle** is the largest possible circle constructed on a Voronoi diagram that contains **no sites**
- The **centre** of the circle will be one of the vertices of a **cell** or **region**
  - The **vertices** of each region are the **intersections** of **the boundaries**
- The **radius** of the circle will be the **distance** from the vertex to the closest site
  - The closest site will be on the circumference
  - Use Pythagoras' Theorem to find the distance
- There may be a **scale** to convert the distance found on the Voronoi diagram into a distance in real life
  - For example if the scale is 1 unit represents 5 km then 5 units represents 25 km

Exam Tip

- The solution to the toxic waste dump will always be one of the points of intersection between the perpendicular bisectors, so you need to know the coordinates of these points
  - Remember that you can use your GDC to solve a pair of the simultaneous equations quickly if you know the equations of two of the perpendicular bisectors that intersect at that point

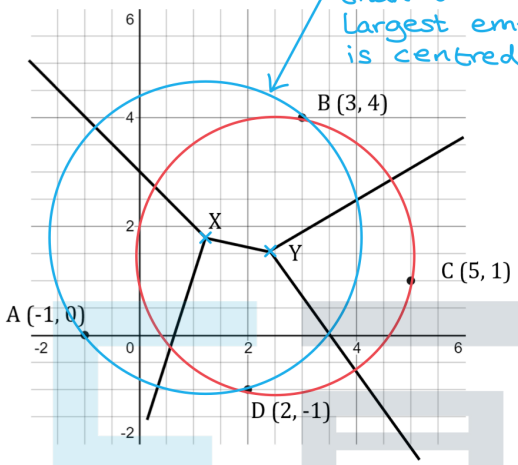
Worked example

The Voronoi diagram below shows four cities at the sites A, B, C and D. The coordinates of the points X and Y are $\left(\frac{5}{4}, \frac{7}{4}\right)$ and $\left(\frac{5}{2}, \frac{3}{2}\right)$ respectively.



Determine the optimal position where a toxic waste site could be located and, given that 1 unit represents 50 km, find the distance from this point to its nearest city.

The optimal position would be at the point X or Y
Draw the Largest possible circle centred at X and Y.



Choose any of the sites on the circle to find the distance

Formula for distance between two points:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$X: (\frac{5}{4}, \frac{7}{4})$ $B: (3, 4)$

Sub coordinates

$$d = \sqrt{(\frac{5}{4} - 3)^2 + (\frac{7}{4} - 4)^2} = \sqrt{(\frac{7}{4})^2 + (\frac{9}{4})^2} = \sqrt{\frac{65}{8}}$$

$= 2.8504\ldots$ units

distance $= 2.8504\ldots \times 50\text{km} = 142.52\ldots$ km

distance = 143 km (3 s.f.)