3.4 Further Trigonometry

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Course	DP IB Maths
Section	3. Geometry & Trigonometry
Topic	3.4 Further Trigonometry
Difficulty	Medium

To be used by all students preparing for DP IB Maths AA SL
Students of other boards may also find this useful

Complete the table.

Question 1

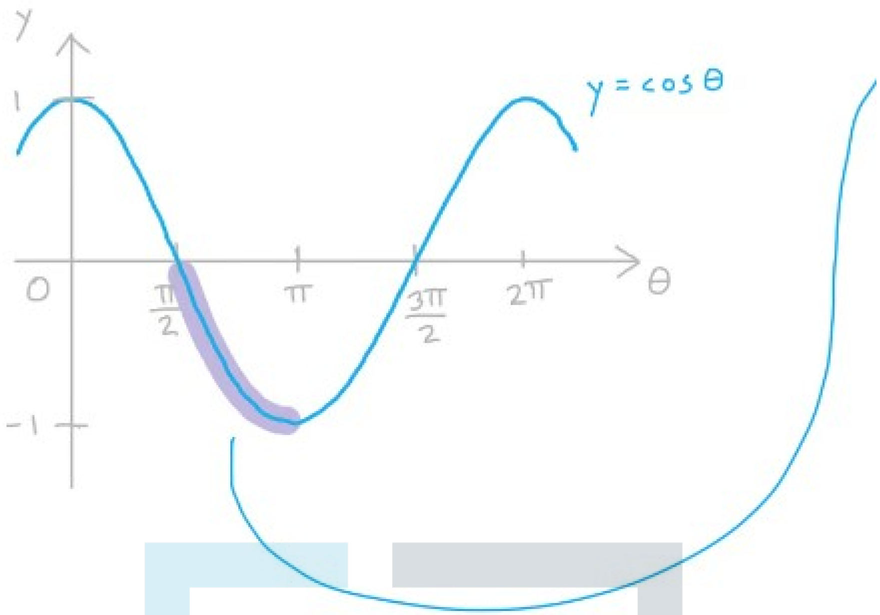
Degrees	Radians	sin	cos	tan
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$120°$	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
$270°$	$\frac{3\pi}{2}$	-1	0	~~ ~~

Notes

① $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ and $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

② The tangent of $270°$ (like the tangent of every multiple of $90°$) is undefined. Sometimes '∞' or '$\pm\infty$' is used to indicate this.

Question 2



$\cos^2\theta + \sin^2\theta = 1$ } Pythagorean identity

$$\cos^2\theta + \left(\frac{3}{5}\right)^2 = \cos^2\theta + \frac{9}{25} = 1$$

$$\cos^2\theta = 1 - \frac{9}{25} = \frac{16}{25}$$

$$\cos\theta = \pm\sqrt{\frac{16}{25}} = \pm\frac{4}{5}$$

But for $\frac{\pi}{2} < \theta < \pi$, $\cos\theta$ is negative so

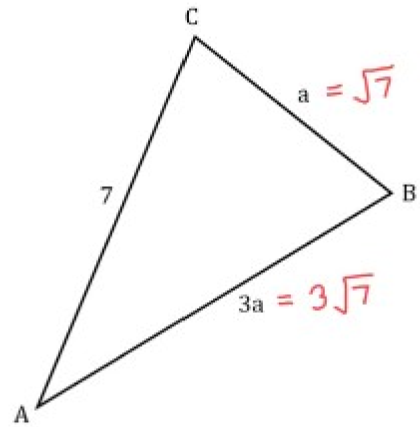
$$\boxed{\cos\theta = -\frac{4}{5}}$$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ } Identity for $\tan\theta$

$$\tan\theta = \frac{3/5}{-4/5} = \frac{3}{5} \times \left(-\frac{5}{4}\right)$$

$$\boxed{\tan\theta = -\frac{3}{4}}$$

Question 3



Given that $\cos A\hat{B}C = \frac{1}{2}$, find the area of the triangle. Give your answer in the form $\frac{p\sqrt{3}}{r}$, where $p, q \in \mathbb{R}$.

$c^2 = a^2 + b^2 - 2ab\cos C$ } Cosine rule

$\cos^2\theta + \sin^2\theta = 1$ } Pythagorean identity

$\text{Area} = \frac{1}{2}ab\sin C$ } area of a triangle

Use cosine rule to find value of a

$$7^2 = (a)^2 + (3a)^2 - 2(a)(3a)\left(\frac{1}{2}\right)$$

$$49 = a^2 + 9a^2 - 3a^2$$

$$7a^2 = 49 \implies a^2 = 7 \implies a = \sqrt{7}$$

Use identity to find $\sin A\hat{B}C$

$$\sin A\hat{B}C = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$$

Now use formula to find area of triangle

$$\text{Area} = \frac{1}{2}(\sqrt{7})(3\sqrt{7})\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2}(21)\left(\frac{\sqrt{3}}{2}\right) = \frac{21\sqrt{3}}{4}$$

$$\boxed{\text{Area} = \frac{21\sqrt{3}}{4} \text{ units}^2}$$

$p = 21$
$r = 4$

Question 4

a) $\cos^2\theta + \sin^2\theta = 1$ } Pythagorean identity

Use identity to find $\sin B\hat{A}C$

$$\left(\frac{2}{3}\right)^2 + \sin^2 B\hat{A}C = 1$$

$$\sin^2 B\hat{A}C + \frac{4}{9} = 1$$

$$\sin^2 B\hat{A}C = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\sin B\hat{A}C = \sqrt{\frac{5}{9}}$$

$$\boxed{\sin B\hat{A}C = \frac{\sqrt{5}}{3}}$$

b) $\text{Area} = \frac{1}{2}ab\sin C$ } area of a triangle

Use formula to find area of triangle

$$\text{Area} = \frac{1}{2}(15)(20)\left(\frac{\sqrt{5}}{3}\right)$$

$$= \frac{1}{2}(300)\left(\frac{\sqrt{5}}{3}\right)$$

$$= \frac{300\sqrt{5}}{6} = 50\sqrt{5}$$

$$\boxed{\text{Area} = 50\sqrt{5} \text{ units}^2}$$

c) $c^2 = a^2 + b^2 - 2ab\cos C$ } Cosine rule

Use cosine rule to find value of x

$$x^2 = (15)^2 + (20)^2 - 2(15)(20)\left(\tfrac{2}{3}\right)$$

$$= 225 + 400 - 600\left(\tfrac{2}{3}\right)$$

$$= 225 + 400 - 400 = 225$$

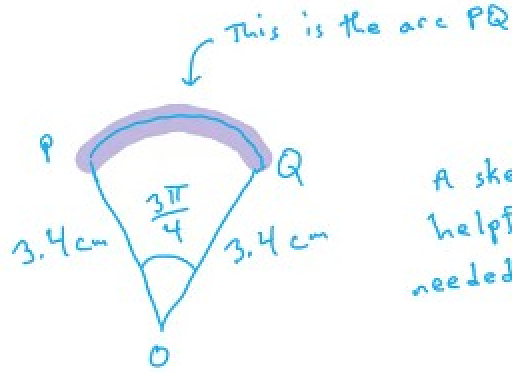
$$\Longrightarrow x = \sqrt{225} = 15$$

$x = 15$, so $AB = BC = 15$.

Two sides are equal, therefore triangle ABC is isosceles.

Question 5

Arc length: $l = r\theta$
Area of sector: $A = \frac{1}{2}r^2\theta$ } θ must be in radians!



A sketch can be helpful, but isn't needed to get the marks

(i) $l = r\theta = (3.4)\left(\frac{3\pi}{4}\right) = \boxed{\frac{51\pi}{20}\text{ cm}}$

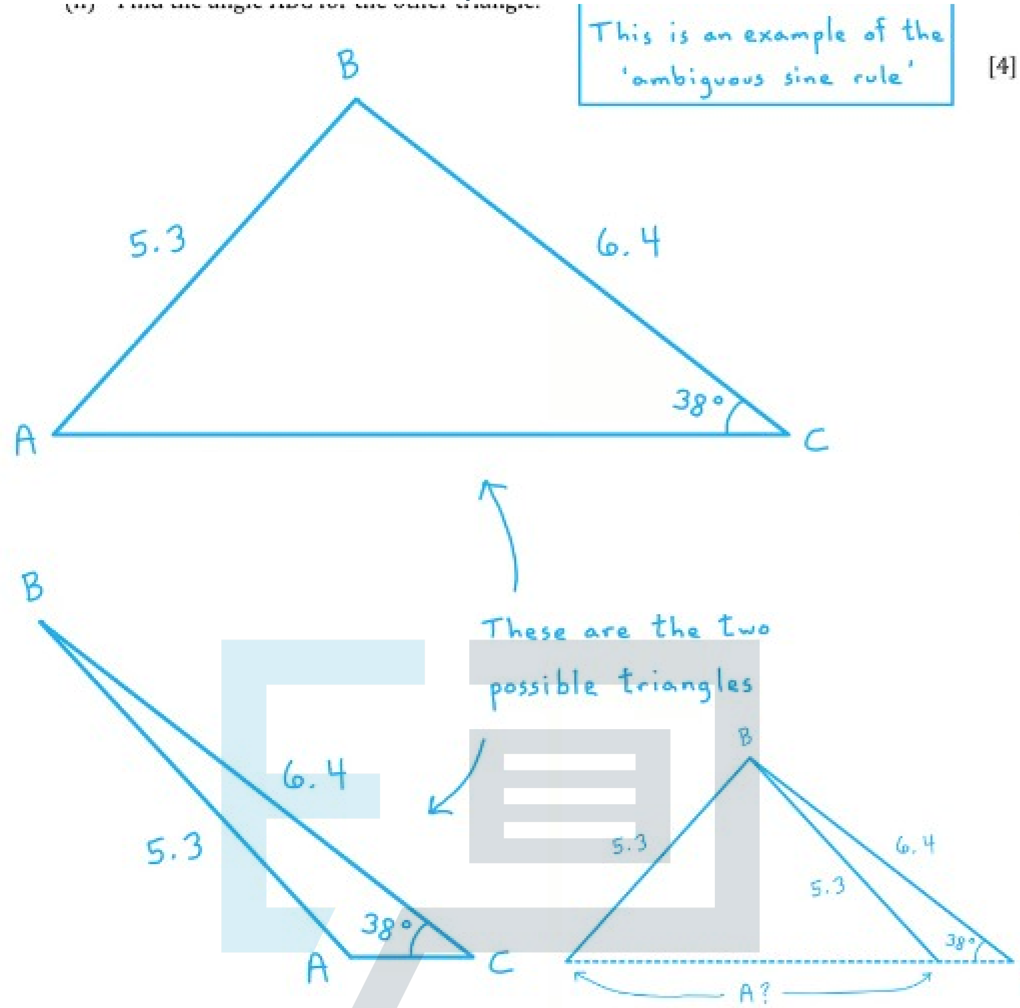
$(\approx 8.0 \text{ cm})$

(ii) $A = \frac{1}{2}r^2\theta = \frac{1}{2}(3.4)^2\left(\frac{3\pi}{4}\right) = \boxed{\frac{867\pi}{200}\text{ cm}^2}$

$(\approx 13.6 \text{ cm}^2)$

(ii) Find the angle ABC for the other triangle.

Question 6



[4]

(i) $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ } Sine rule

$$\frac{6.4}{\sin B\hat{A}C} = \frac{5.3}{\sin 38°} \implies \sin B\hat{A}C = \frac{6.4}{5.3}\sin 38°$$

$$B\hat{A}C = \sin^{-1}\left(\frac{6.4}{5.3}\sin 38°\right) = 48.025304...°$$

$\sin\theta = \sin(180° - \theta)$ [Property of sine function]*

$$\text{or } B\hat{A}C = 180 - \sin^{-1}\left(\frac{6.4}{5.3}\sin 38°\right) = 131.974695...°$$

$$\boxed{B\hat{A}C = 132° \ (3 \text{ s.f.})}$$

* This can be seen in the symmetry of the sine graph.

(ii) In the other triangle,

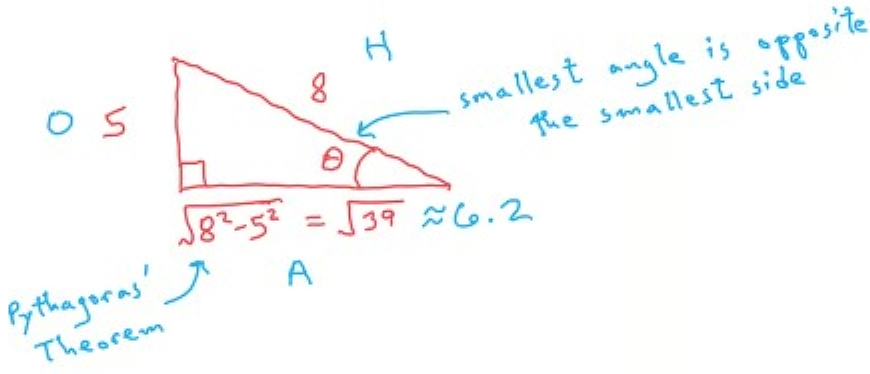
$$B\hat{A}C = \sin^{-1}\left(\frac{6.4}{5.3}\sin 38°\right) = 48.025304...°$$

Therefore

$$A\hat{B}C = 180 - 38 - \sin^{-1}\left(\frac{6.4}{5.3}\sin 38°\right) = 93.974695...°$$

$$\boxed{A\hat{B}C = 94.0° \ (3 \text{ s.f.})}$$

Question 7



$$\boxed{\begin{aligned} \sin\theta &= \frac{5}{8} \\ \cos\theta &= \frac{\sqrt{39}}{8} \\ \tan\theta &= \frac{5}{\sqrt{39}} \end{aligned}}$$

SOH

CAH

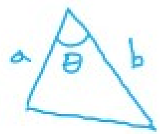
TOA

Question 8

Arc length : $\ell = r\theta$

Area of sector : $A = \frac{1}{2}r^2\theta$

θ must be in radians!



Area $= \frac{1}{2}ab\sin\theta$

a) (i) area $= \frac{1}{2}(5.4)^2(1.2) = 17.496 \text{ cm}^2$

$\boxed{= 17.5 \text{ cm}^2 \text{ (3 s.f.)}}$

(ii) area $= \frac{1}{2}(5.4)(5.4)\sin(1.2) = 13.589\ldots \text{ cm}^2$

$\boxed{= 13.6 \text{ cm}^2 \text{ (3 s.f.)}}$

(iii) area $= 17.496 - 13.589\ldots$

$= 3.906\ldots \text{ cm}^2$

$\boxed{= 3.91 \text{ cm}^2 \text{ (3 s.f.)}}$

Arc length : $\ell = r\theta$

Area of sector : $A = \frac{1}{2}r^2\theta$

θ must be in radians!

b) (i) $\ell = (5.4)(1.2) = \boxed{6.48 \text{ cm}}$

(ii) Perimeter $= 5.4 + 5.4 + 6.48$

$\boxed{= 17.28 \text{ cm}}$

Question 9

Arc length : $l = r\theta$

Area of sector : $A = \frac{1}{2} r^2 \theta$

θ must be in radians!

$$\text{area} = \frac{81\pi}{200} = \frac{1}{2} r^2 \left(\frac{\pi}{4}\right)$$

$$\frac{\cancel{\pi}}{8} r^2 = \frac{81\cancel{\pi}}{200}$$

$$\frac{1}{8} r^2 = \frac{81}{200}$$

$$r^2 = \frac{81}{25}$$

$$\text{radius} = \sqrt{\frac{81}{25}} = \frac{9}{5} \text{ m}$$

$$\boxed{\text{length of connecting cord is } \frac{9}{5} \text{ m } (1.8 \text{ m})}$$

Question 10

Arc length : $l = r\theta$

Area of sector : $A = \frac{1}{2} r^2 \theta$

θ must be in radians!

(i) $\text{Area} = \frac{1}{2}(8)^2(1.2) = \boxed{38.4 \text{ cm}^2}$

(ii) $\text{Volume} = 38.4 \times 2 = \boxed{76.8 \text{ cm}^3}$

Volume of Prism = Area of Cross-section × Height

Question 11

Arc length: $l = r\theta$

Area of sector: $A = \frac{1}{2} r^2 \theta$

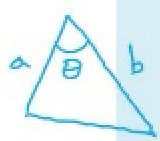
θ must be in radians!

a)

radius of sector OAB $= (p+8)$ cm

$$\text{Area} = \frac{1}{2}(p+8)^2\left(\frac{\pi}{3}\right)$$

$$= \frac{\pi}{6}(p+8)^2 \text{ cm}^2$$



Area $= \frac{1}{2} ab \sin\theta$

b)

$$\text{Area} = \frac{1}{2}(8)(8)\sin\left(\frac{\pi}{3}\right)$$

$$= 32\sin\left(\frac{\pi}{3}\right)$$

$$= 32\left(\frac{\sqrt{3}}{2}\right)$$

$$= 16\sqrt{3} \text{ cm}^2$$

c) area of shape ABCD = area of sector OAB − area of triangle OCD

$$= \frac{\pi}{6}(p+8)^2 - 16\sqrt{3}$$

Therefore

$$\frac{\pi}{6}(p+8)^2 \cancel{-16\sqrt{3}} = \frac{50\pi}{3} \cancel{-16\sqrt{3}}$$

$$\frac{\cancel{\pi}}{6}(p+8)^2 = \frac{50\cancel{\pi}}{3}$$

$$\frac{1}{6}(p+8)^2 = \frac{50}{3}$$

$$(p+8)^2 = 100$$

$$p+8 = \pm 10$$

$$p = 2 \text{ or } -18$$

But p can't be negative

because it is the length of a line segment!

$$\boxed{\text{So } p = 2 \text{ cm}}$$