DP IB Maths: AA SL

3.4 Further Trigonometry

Contents

3.4.1 The Unit Circle

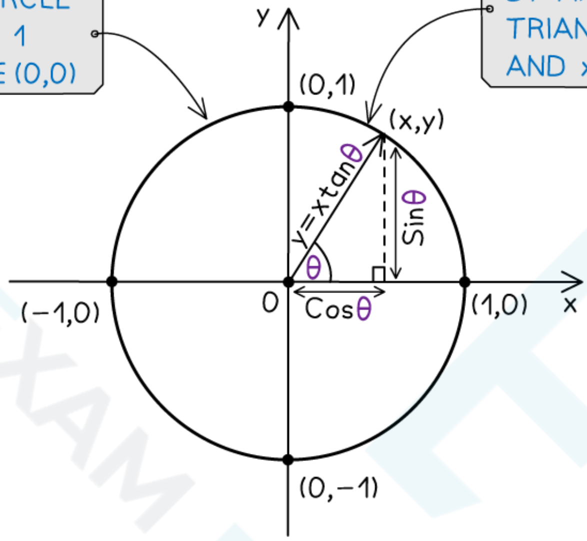
Defining Sin, Cos and Tan

What is the unit circle?

- The unit circle is a circle with radius 1 and centre (0, 0)
- Angles are always measured from the positive x-axis and turn:
  - **anticlockwise** for **positive** angles
  - **clockwise** for **negative** angles
- It can be used to calculate trig values as a coordinate point (x, y) on the circle
  - Trig values can be found by making a right triangle with the radius as the hypotenuse
  - Where θ is the angle measured anticlockwise from the positive *x*-axis
  - The *x*-axis will always be adjacent to the angle, θ
- SOHCAHTOA can be used to find the values of sinθ, cosθ and tanθ easily
- As the radius is 1 unit
  - the ***x* coordinate** gives the value of **cosθ**
  - the ***y* coordinate** gives the value of **sinθ**
- As the origin is one of the end points - dividing the y coordinate by the x coordinate gives the gradient
  - the **gradient** of the line gives the value of **tanθ**
- It allows us to calculate sin, cos and tan for angles greater than 90° ($\frac{\pi}{2}$ rad)

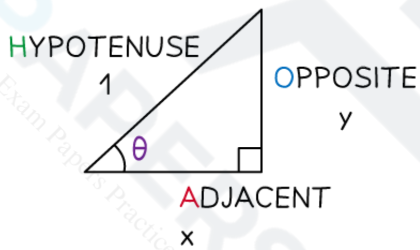
THE UNIT CIRCLE HAS RADIUS 1 AND CENTRE (0,0)

TRIG VALUES CAN BE FOUND BY MAKING A RIGHT ANGLED TRIANGLE WITH THE RADIUS AND x−AXIS



THE ANGLE IS MEASURED ANTI−CLOCKWISE FROM THE X−AXIS IN EITHER RADIANS OR DEGREES

ANY POINT (x,y) ON THE UNIT CIRCLE CAN BE FOUND USING (cosθ, sinθ)



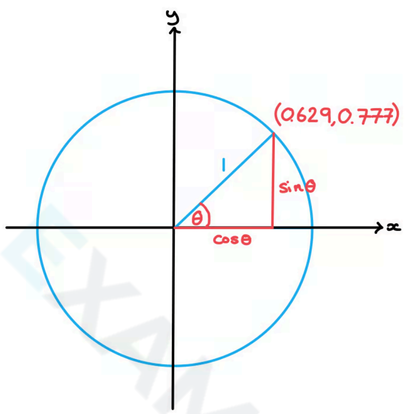
$$\text{Cos}\theta = \frac{A}{H} = \frac{x}{1} = x$$

$$\text{Sin}\theta = \frac{O}{H} = \frac{y}{1} = y$$

$$\text{Tan}\theta = \frac{O}{A} = \frac{y}{x}$$

Worked example

The coordinates of a point on a unit circle, to 3 significant figures, are (0.629, 0.777). Find θ° to the nearest degree.



We know $(x, y) = (\cos\theta, \sin\theta)$

so,

$\cos\theta = 0.629$

$\sin\theta = 0.777$

Using either ratio :

$\theta = \cos^{-1}(0.629)$

$= 51.023...$

$\theta = 51°$ (nearest degree)

Using The Unit Circle

What are the properties of the unit circle?

- The unit circle can be split into four **quadrants** at every 90° ($\frac{\pi}{2}$ rad)
  - The first quadrant is for angles between 0 and 90°
    - **A**ll three of Sinθ, Cosθ and Tanθ are positive in this quadrant
  - The second quadrant is for angles between 90° and 180° ($\frac{\pi}{2}$ rad and π rad)
    - **S**inθ is positive in this quadrant
  - The third quadrant is for angles between 180° and 270° (π rad and $\frac{3\pi}{2}$)
    - **T**anθ is positive in this quadrant
  - The fourth quadrant is for angles between 270° and 360° ($\frac{3\pi}{2}$ rad and 2π)
    - **C**osθ is positive in this quadrant
  - Starting from the **fourth** quadrant (on the bottom right) and working anti-clockwise the positive trig functions spell out **CAST**
    - This is why it is often thought of as the **CAST** diagram
    - You may have your own way of remembering this
    - A popular one starting from the first quadrant is **All Students Take Calculus**
  - To help picture this better try sketching all three trig graphs on one set of axes and look at which graphs are positive in each 90° section

How is the unit circle used to find secondary solutions?

- Trigonometric functions have more than one input to each output
  - For example sin 30° = sin 150° = 0.5
  - This means that trigonometric equations have more than one solution
  - For example both 30° and 150° satisfy the equation sin x = 0.5
- The unit circle can be used to find all solutions to trigonometric equations in a given interval
  - Your calculator will only give you the first solution to a problem such as $x = \sin^{-1}(0.5)$
    - This solution is called the **primary value**
  - However, due to the **periodic** nature of the trig functions there could be an infinite number of solutions
    - Further solutions are called the **secondary values**
  - This is why you will be given a **domain** in which your solutions should be found
    - This could either be in degrees or in radians
    - If you see π or some multiple of π then you must work in radians
- The following steps may help you use the unit circle to find **secondary values**

STEP 1: Draw the angle into the first quadrant using the x or y coordinate to help you

- If you are working with sin x = k, draw the line from the origin to the circumference of the circle at the point where the **y coordinate** is k

- If you are working with cos $x = k$, draw the line from the origin to the circumference of the circle at the point where the **x coordinate** is k
- If you are working with tan $x = k$, draw the line from the origin to the circumference of the circle such that the gradient of the line is k
  - This will give you the angle which should be measured from the **positive x-axis…**
    - … anticlockwise for a positive angle
    - … clockwise for a negative angle

STEP 2: Draw the radius in the other quadrant which has the same...

- ... x-coordinate if solving cos $x = k$
  - This will be the quadrant which is vertical to the original quadrant
- ... y-coordinate if solving sin x = k
  - This will be the quadrant which is horizontal to the original quadrant
- ... gradient if solving tan x = k
  - This will be the quadrant diagonal to the original quadrant

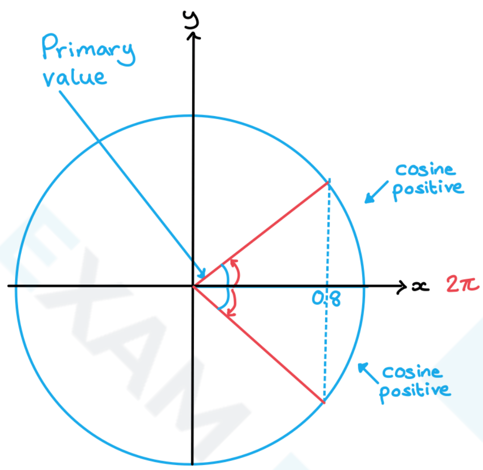
STEP 3: Work out the size of the second angle, measuring from the positive x-axis

- … anticlockwise for a positive angle
- … clockwise for a negative angle
  - You should look at the given range of values to decide whether you need the negative or positive angle

STEP 4: Add or subtract either 360° or 2π radians to both values until you have all solutions in the required range

Worked example

Given that one solution of $\cos\theta = 0.8$ is $\theta = 0.6435$ radians correct to 4 decimal places, find all other solutions in the range $-2\pi \leq \theta \leq 2\pi$. Give your answers correct to 3 significant figures.



Cosine is positive in the first and fourth quadrants so draw the angle from the horizontal axis in both quadrants.

Primary value $= 0.6435$

Using diagram, secondary value $= -0.6435$

Therefore all values are: $0.6435 \pm 2\pi n$

and $-0.6435 \pm 2\pi n$

Within given domain: $-2\pi \leq \theta \leq 2\pi$

$\theta = -5.64^c, -0.644^c, 0.644^c, 5.64^c$

3.4.2 Exact Values

Trigonometry Exact Values

What are exact values in trigonometry?

- For certain angles the values of sin θ, cos θ and tan θ can be written **exactly**
  - This means using fractions and surds
  - You should be familiar with these values and be able to derive the values using geometry
- You are expected to know the exact values of sin, cos and tan for angles of 0°, 30°, 45°, 60°, 90°, 180° and their multiples
  - In **radians** this is $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$ and their multiples
- The exact values you are expected to know are here:

Angle (degrees)	**0**	**30**	**45**	**60**	**90**	**120**	**135**	**150**	**180**	**210**	**225**	**240**	**270**	**300**	**315**	**330**	**360**
Angle (Radians)	**0**	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	UD	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	UD	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0
csc	UD	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	UD	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1	$-\frac{2}{\sqrt{3}}$	$-\sqrt{2}$	-2	UD
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	UD	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1	$-\frac{2}{\sqrt{3}}$	$-\sqrt{2}$	-2	UD	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
cot	UD	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	UD	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$-\sqrt{3}$	UD

UD -> Undefined

How do I find the exact values of other angles?

- The exact values for sin and cos can be seen on the **unit circle** as the *y* and *x* coordinates respectively
  - If using the coordinates on the unit circle to memorise the exact values, remember that cos comes **before** sin
- The **unit circle** can also be used to find exact values of other angles using symmetry
- If you know the exact value for an angle in the first quadrant you can draw the same angle from the x-axis in any other quadrant to find other angles
- Remember that the angles are **measured anticlockwise** from the positive x-axis
- For example if you know that the exact value for is 0.5

- draw the angle 30° from the horizontal in the three other quadrants
- measuring from the positive x-axis you have the angles of 150°, 210° and 330°
  - sin is positive in the second quadrant so sin150° = 0.5
  - sin is negative in the third quadrant so sin210° = - 0.5
  - sin is negative in the fourth quadrant so sin330° = - 0.5

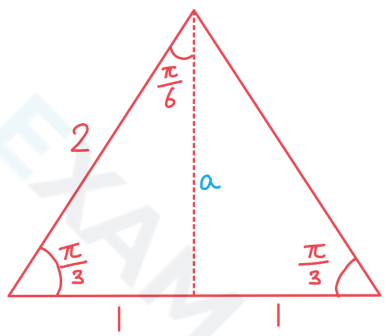
- It is also possible to find the **negative** angles by measuring **clockwise** from the positive x-axis
  - draw the angle 30° from the horizontal in the three other quadrants
  - measuring **clockwise** from the positive x-axis you have the angles of -30°, -150°, -210° and -330°
    - sin is negative in the fourth quadrant so sin(-30°) = - 0.5
    - sin is negative in the third quadrant so sin(-150°) = - 0.5
    - sin is positive in the second quadrant so sin(-210°) = 0.5
    - sin is positive in the fourth quadrant so sin(-330°) = 0.5

How are exact values in trigonometry derived?

- There are two special **right-triangles** that can be used to derive all of the exact values you need to know
- Consider a **right-triangle** with a hypotenuse of 2 units and a shorter side length of 1 unit
  - Using Pythagoras' theorem the third side will be $\sqrt{3}$
  - The angles will be $\frac{\pi}{2}$ radians (90°), $\frac{\pi}{3}$ radians (60°) and $\frac{\pi}{6}$ radians (30°)
  - Using SOHCAHTOA gives...
    - Sin $\frac{\pi}{3}$ = $\frac{\sqrt{3}}{2}$ Sin $\frac{\pi}{6}$ = $\frac{1}{2}$
    - Cos $\frac{\pi}{3}$ = $\frac{1}{2}$ Cos $\frac{\pi}{6}$ = $\frac{\sqrt{3}}{2}$
    - Tan $\frac{\pi}{3}$ = $\sqrt{3}$ Tan $\frac{\pi}{6}$ = $\frac{1}{\sqrt{3}}$ = $\frac{\sqrt{3}}{3}$
- Consider an **isosceles triangle** with two equal side lengths (the opposite and adjacent) of 1 unit
  - Using Pythagoras' theorem it will have a hypotenuse of $\sqrt{2}$
  - The two equal angles will be $\frac{\pi}{4}$ radians (45°)
  - Using SOHCAHTOA gives...
    - Sin $\frac{\pi}{4}$ = $\frac{1}{\sqrt{2}}$ = $\frac{\sqrt{2}}{2}$
    - Cos $\frac{\pi}{4}$ = $\frac{1}{\sqrt{2}}$ = $\frac{\sqrt{2}}{2}$
    - Tan $\frac{\pi}{4}$ = 1

Worked example

Using an equilateral triangle of side length 2 units, derive the exact values for the sine, cosine and tangent of $\dfrac{\pi}{6}$ and $\dfrac{\pi}{3}$.



Use Pythagoras' Theorem to find a :

$$a^2 = \sqrt{2^2 - 1^2}$$
$$= \sqrt{3}$$

Using SOHCAH TOA :

$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$ $\tan\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{1} = \sqrt{3}$