EXAM PAPERS PRACTICE

3.4 Further Trigonometry

IB Maths - Revision Notes

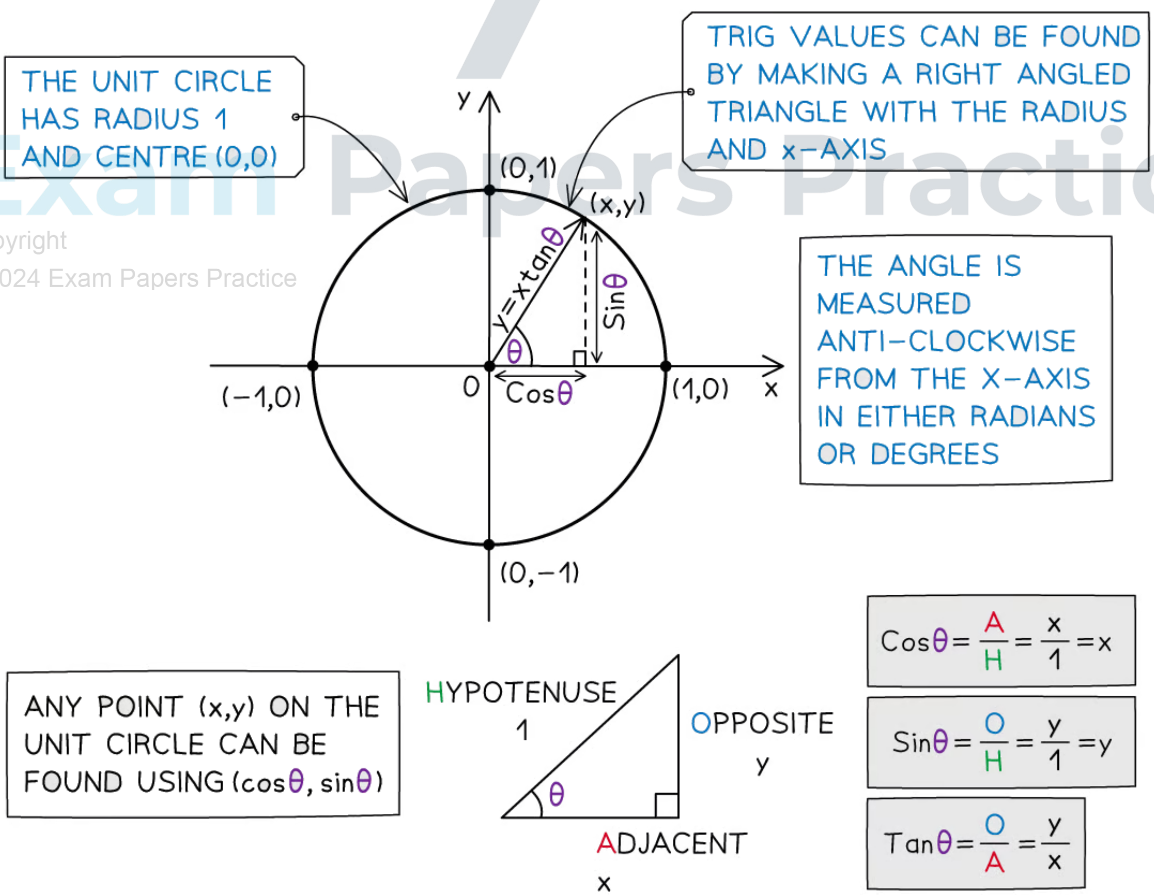
AA SL

3.4.1 The Unit Circle

Defining Sin, Cos and Tan

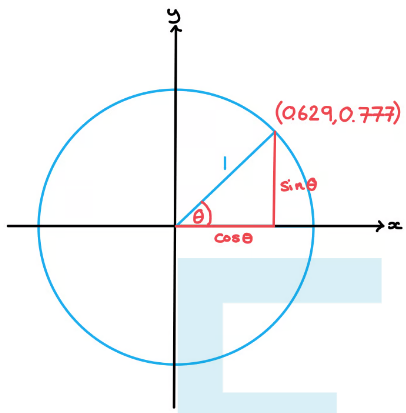
What is the unit circle?

- The unit circle is a circle with radius 1 and centre (0, 0)
- Angles are always measured from the positive x-axis and turn:
  - **anticlockwise** for **positive** angles
  - **clockwise** for **negative** angles
- It can be used to calculate trig values as a coordinate point (x, y) on the circle
  - Trig values can be found by making a right triangle with the radius as the hypotenuse
  - Where θ is the angle measured anticlockwise from the positive *x*-axis
  - The *x*-axis will always be adjacent to the angle, θ
- SOHCAHTOA can be used to find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ easily
- As the radius is 1 unit
  - the ***x* coordinate** gives the value of $\cos\theta$
  - the ***y* coordinate** gives the value of $\sin\theta$
- As the origin is one of the end points - dividing the y coordinate by the x coordinate gives the gradient
  - the **gradient** of the line gives the value of $\tan\theta$
- It allows us to calculate sin, cos and tan for angles greater than 90° ($\frac{\pi}{2}$ rad)



Worked example

The coordinates of a point on a unit circle, to 3 significant figures, are (0.629, 0.777). Find θ° to the nearest degree.



We know $(x, y) = (\cos\theta, \sin\theta)$

So,

$\cos\theta = 0.629$

$\sin\theta = 0.777$

Using either ratio:

$\theta = \cos^{-1}(0.629)$

$= 51.023\ldots$

$\boxed{\theta = 51° \text{ (nearest degree)}}$

Using The Unit Circle

What are the properties of the unit circle?

- The unit circle can be split into four **quadrants** at every 90° ($\frac{\pi}{2}$ rad)
  - The first quadrant is for angles between 0 and 90°
    - **A**ll three of Sinθ, Cosθ and Tanθ are positive in this quadrant
  - The second quadrant is for angles between 90° and 180° ($\frac{\pi}{2}$ rad and π rad)
    - **S**inθ is positive in this quadrant
  - The third quadrant is for angles between 180° and 270° (π rad and $\frac{3\pi}{2}$)
    - **T**anθ is positive in this quadrant
  - The fourth quadrant is for angles between 270° and 360° ($\frac{3\pi}{2}$ rad and 2π)
    - **C**osθ is positive in this quadrant
  - Starting from the **fourth** quadrant (on the bottom right) and working anti-clockwise the positive trig functions spell out **CAST**
    - This is why it is often thought of as the **CAST** diagram
    - You may have your own way of remembering this
    - A popular one starting from the first quadrant is **All Students Take Calculus**
  - To help picture this better try sketching all three trig graphs on one set of axes and look at which graphs are positive in each 90° section

How is the unit circle used to find secondary solutions?

- Trigonometric functions have more than one input to each output
  - For example sin 30° = sin 150° = 0.5
  - This means that trigonometric equations have more than one solution
  - For example both 30° and 150° satisfy the equation sin x = 0.5
- The unit circle can be used to find all solutions to trigonometric equations in a given interval
  - Your calculator will only give you the first solution to a problem such as $x = \sin^{-1}(0.5)$
    - This solution is called the **primary value**
  - However, due to the **periodic** nature of the trig functions there could be an infinite number of solutions
    - Further solutions are called the **secondary values**
  - This is why you will be given a **domain** in which your solutions should be found
    - This could either be in degrees or in radians
    - If you see π or some multiple of π then you must work in radians
- The following steps may help you use the unit circle to find **secondary values**

STEP 1: Draw the angle into the first quadrant using the x or y coordinate to help you

- If you are working with sin $x = k$, draw the line from the origin to the circumference of the circle at the point where the **y coordinate** is k
- If you are working with cos $x = k$, draw the line from the origin to the circumference of the circle at the point where the **x coordinate** is k
- If you are working with tan $x = k$, draw the line from the origin to the circumference of the circle such that the gradient of the line is k
  - This will give you the angle which should be measured from the **positive x-axis…**
    - …anticlockwise for a positive angle
    - …clockwise for a negative angle

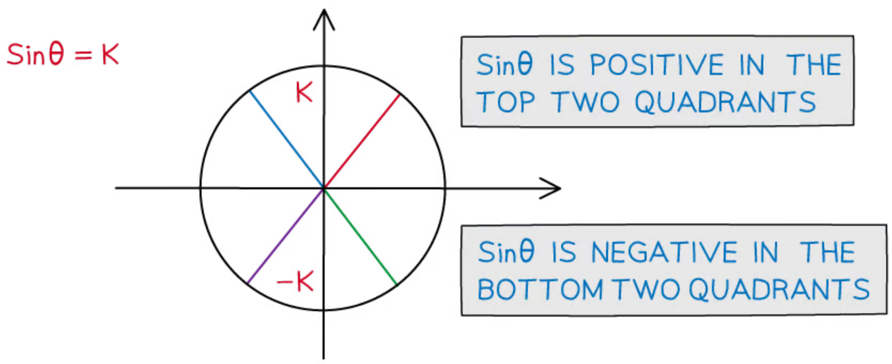
STEP 2: Draw the radius in the other quadrant which has the same…

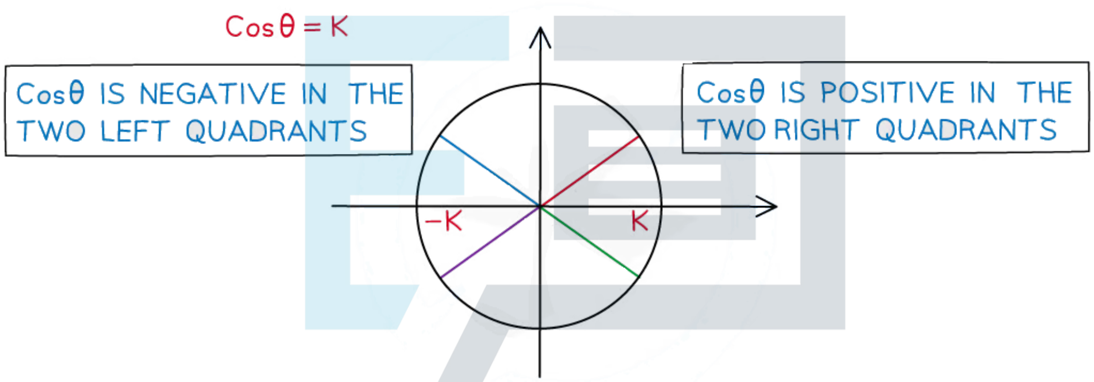
- … x-coordinate if solving cos $x = k$
  - This will be the quadrant which is vertical to the original quadrant
- … y-coordinate if solving sin $x = $ k
  - This will be the quadrant which is horizontal to the original quadrant
- … gradient if solving tan x = k
  - This will be the quadrant diagonal to the original quadrant

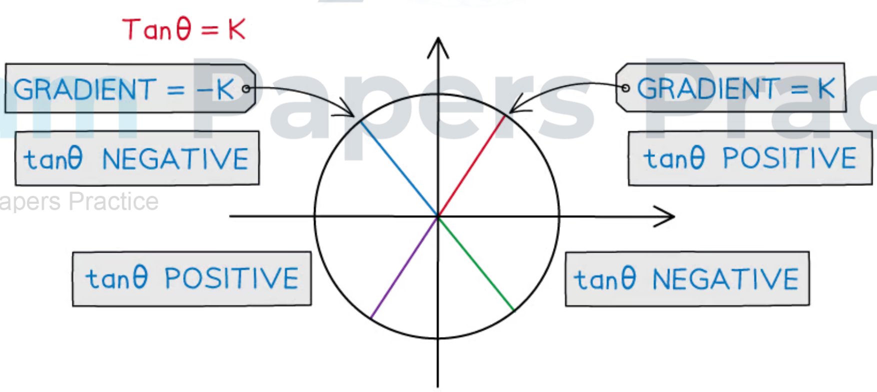
STEP 3: Work out the size of the second angle, measuring from the positive x-axis

- …anticlockwise for a positive angle
- …clockwise for a negative angle
  - You should look at the given range of values to decide whether you need the negative or positive angle

STEP 4: Add or subtract either 360° or 2π radians to both values until you have all solutions in the required range





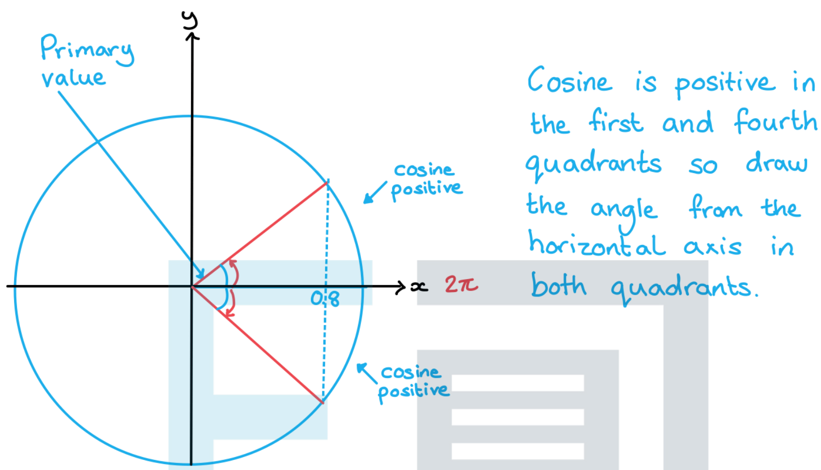


Exam Tip

- Being able to sketch out the unit circle and remembering CAST can help you to find all solutions to a problem in an exam question

Worked example

Given that one solution of $\cos\theta = 0.8$ is $\theta = 0.6435$ radians correct to 4 decimal places, find all other solutions in the range $-2\pi \le \theta \le 2\pi$. Give your answers correct to 3 significant figures.



Primary value $= 0.6435$

Using diagram, secondary value $= -0.6435$

Therefore all values are: $0.6435 \pm 2\pi n$

and $-0.6435 \pm 2\pi n$

Within given domain: $-2\pi \le \theta \le 2\pi$

$$\boxed{\theta = -5.64^c,\ -0.644^c,\ 0.644^c,\ 5.64^c}$$

3.4.2 Exact Values

Trigonometry Exact Values

What are exact values in trigonometry?

- For certain angles the values of sin θ, cos θ and tan θ can be written **exactly**
  - This means using fractions and surds
  - You should be familiar with these values and be able to derive the values using geometry
- You are expected to know the exact values of sin, cos and tan for angles of 0°, 30°, 45°, 60°, 90°, 180° and their multiples
  - In **radians** this is $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$ and their multiples
- The exact values you are expected to know are here:

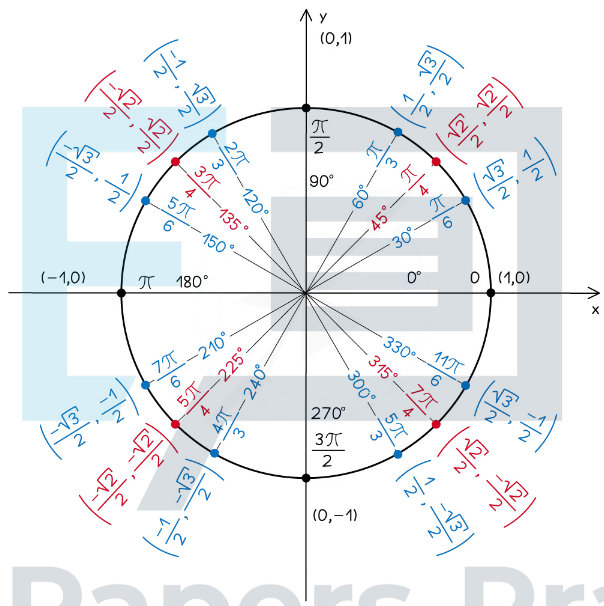
DEGREES	0°	30°	45°	60°	90°	180°	360°
RADIANS	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	2π
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	0
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	−1	1
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	UNDEFINED	0	0

How do I find the exact values of other angles?

- The exact values for sin and cos can be seen on the **unit circle** as the *y* and *x* coordinates respectively
  - If using the coordinates on the unit circle to memorise the exact values, remember that cos comes **before** sin
- The **unit circle** can also be used to find exact values of other angles using symmetry
- If you know the exact value for an angle in the first quadrant you can draw the same angle from the x-axis in any other quadrant to find other angles
- Remember that the angles are **measured anticlockwise** from the positive x-axis
- For example if you know that the exact value for is 0.5
  - draw the angle 30° from the horizontal in the three other quadrants
  - measuring from the positive x-axis you have the angles of 150°, 210° and 330°
    - sin is positive in the second quadrant so sin150° = 0.5
    - sin is negative in the third quadrant so sin210° = - 0.5

- sin is negative in the fourth quadrant so sin330° = - 0.5

- It is also possible to find the **negative** angles by measuring **clockwise** from the positive x-axis
  - draw the angle 30° from the horizontal in the three other quadrants
  - measuring **clockwise** from the positive x-axis you have the angles of -30°, -150°, -210° and -330°
    - sin is negative in the fourth quadrant so sin(-30°) = - 0.5
    - sin is negative in the third quadrant so sin(-150°) = - 0.5
    - sin is positive in the second quadrant so sin(-210°) = 0.5
    - sin is positive in the fourth quadrant so sin(-330°) = 0.5

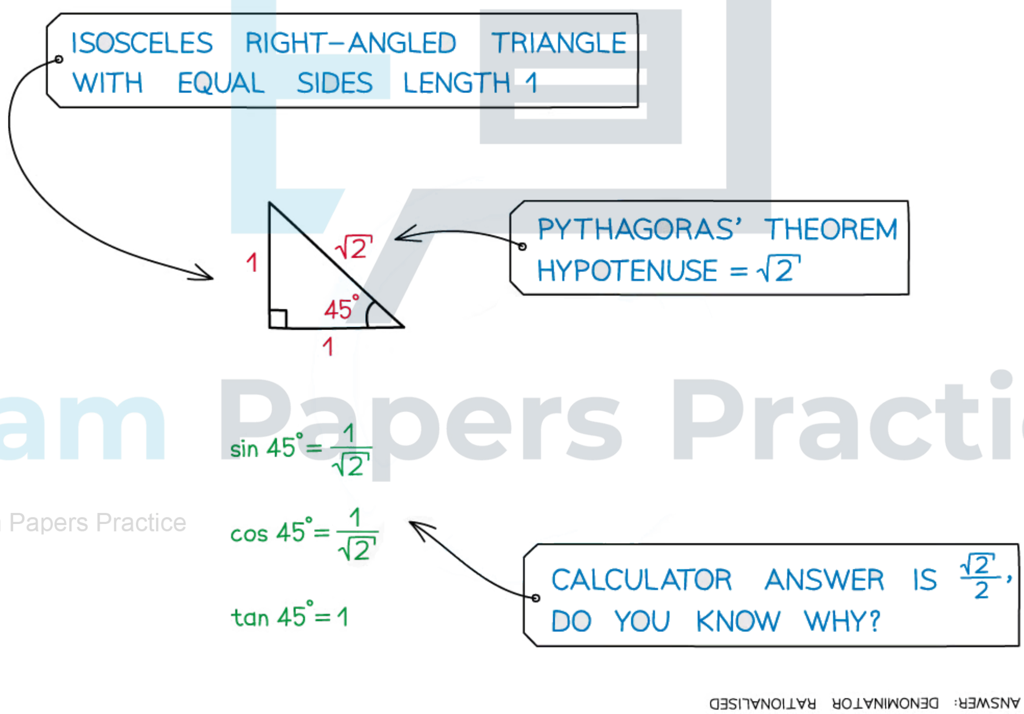


How are exact values in trigonometry derived?

- There are two special **right-triangles** that can be used to derive all of the exact values you need to know
- Consider a **right-triangle** with a hypotenuse of 2 units and a shorter side length of 1 unit
  - Using Pythagoras' theorem the third side will be $\sqrt{3}$
  - The angles will be $\frac{\pi}{2}$ radians (90°), $\frac{\pi}{3}$ radians (60°) and $\frac{\pi}{6}$ radians (30°)
  - Using SOHCAHTOA gives…
    - $\text{Sin}\ \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ $\quad$ $\text{Sin}\ \frac{\pi}{6} = \frac{1}{2}$
    - $\text{Cos}\ \frac{\pi}{3} = \frac{1}{2}$ $\quad$ $\text{Cos}\ \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

- $\text{Tan}\ \dfrac{\pi}{3} = \sqrt{3}$ $\quad\quad$ $\text{Tan}\ \dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

- Consider an **isosceles triangle** with two equal side lengths (the opposite and adjacent) of 1 unit
  - Using Pythagoras' theorem it will have a hypotenuse of $\sqrt{2}$
  - The two equal angles will be $\dfrac{\pi}{4}$ radians (45°)
  - Using SOHCAHTOA gives…
    - $\text{Sin}\ \dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$
    - $\text{Cos}\ \dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$
    - $\text{Tan}\ \dfrac{\pi}{4} = 1$

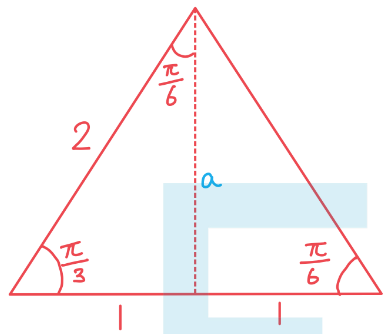


Exam Tip

- You will be expected to be comfortable using exact trig values for certain angles but it can be easy to muddle them up if you just try to remember them from a list, sketch the triangles and trig graphs on your paper so that you can use them as many times as you need to during the exam!
  - sketch the triangles for the key angles $45°/\dfrac{\pi}{4}, 30°/\dfrac{\pi}{6}, 60°/\dfrac{\pi}{3}$
  - sketch the trig graphs for the key angles $0°, 90°/\dfrac{\pi}{2}, 180°/\pi, 270°/\dfrac{3\pi}{2}, 360°/2\pi$

Worked example

Using an equilateral triangle of side length 2 units, derive the exact values for the sine, cosine and tangent of $\dfrac{\pi}{6}$ and $\dfrac{\pi}{3}$.



Use Pythagoras' Theorem to find a:

$$a^2 = \sqrt{2^2 - 1^2}$$
$$= \sqrt{3}$$

Using SOHCAHTOA:

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \qquad \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \qquad \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

$$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}} \qquad \tan\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{1} = \sqrt{3}$$