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3.4 Further Trigonometry

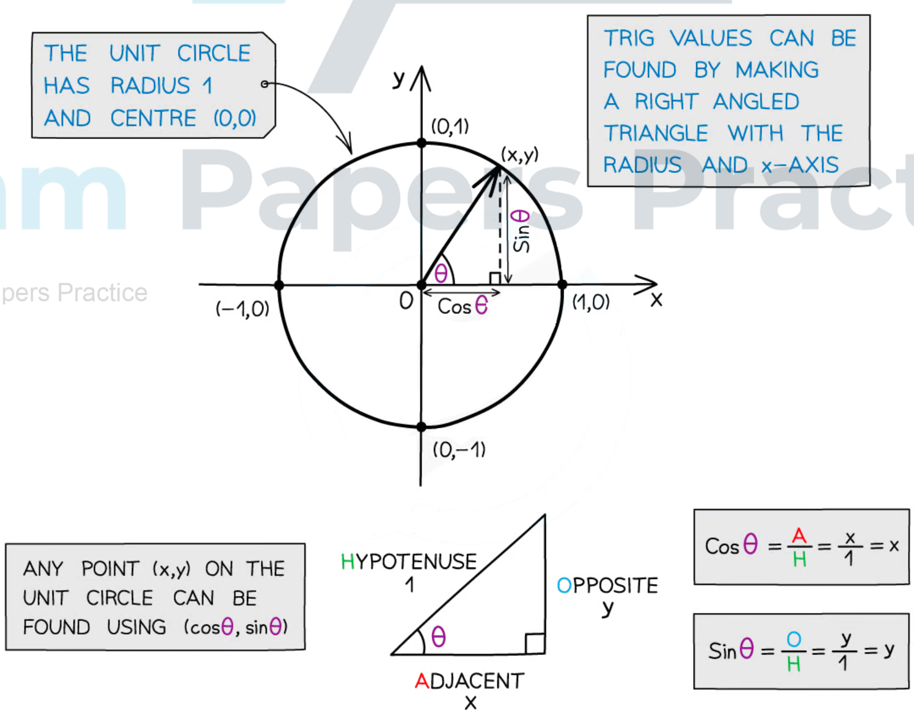
IB Maths - Revision Notes

3.4.1 The Unit Circle

Defining Sin, Cos and Tan

What is the unit circle?

- The unit circle is a circle with radius 1 and centre (0, 0)
- Angles are always measured from the positive x-axis and turn:
  - **anticlockwise** for **positive** angles
  - **clockwise** for **negative** angles
- It can be used to calculate trig values as a coordinate point (x, y) on the circle
  - Trig values can be found by making a right triangle with the radius as the hypotenuse
  - θ is the angle measured anticlockwise from the positive *x*-axis
  - The *x*-axis will always be adjacent to the angle, θ
- SOHCAHTOA can be used to find the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ easily
- As the radius is 1 unit
  - the ***x* coordinate** gives the value of **cosθ**
  - the ***y* coordinate** gives the value of **sinθ**
- As the origin is one of the end points - dividing the y coordinate by the x coordinate gives the gradient
  - the **gradient** of the line gives the value of **tanθ**
- It allows us to calculate sin, cos and tan for angles greater than 90° ($\frac{\pi}{2}$ rad)



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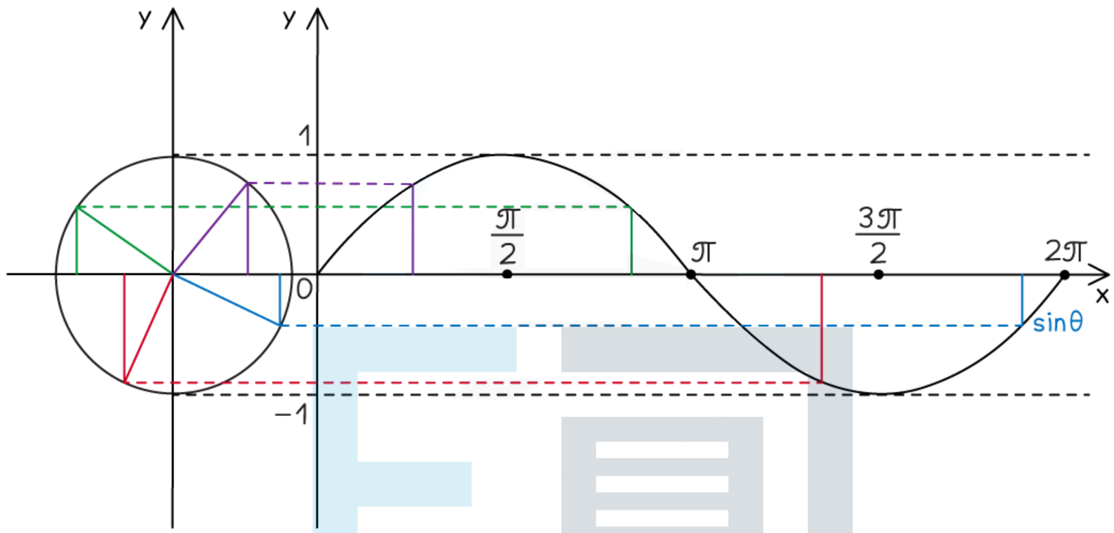
How is the unit circle used to construct the graphs of sine and cosine?

- On the unit circle the ***y*-coordinates** give the value of **sine**
  - Plot the *y*-coordinate from the unit circle as the *y*-coordinate on a trig graph for *x*-coordinates of $\theta = 0, \pi/2, \pi, 3\pi/2$ and 2π
  - Join these points up using a smooth curve
    - To get a clearer idea of the shape of the curve the points for *x*-coordinates of $\theta = \pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$ could also be plotted



- On the unit circle the ***x*-coordinates** give the value of **cosine**
  - Plot the *x*-coordinate from the unit circle as the *y*-coordinate on a trig graph for *x*-coordinates of $\theta = 0, \pi/4, \pi/2, 3\pi/4$ and 2π
  - Join these points up using a smooth curve
    - To get a clearer idea of the shape of the curve the points for *x*-coordinates of $\theta = \pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$ could also be plotted

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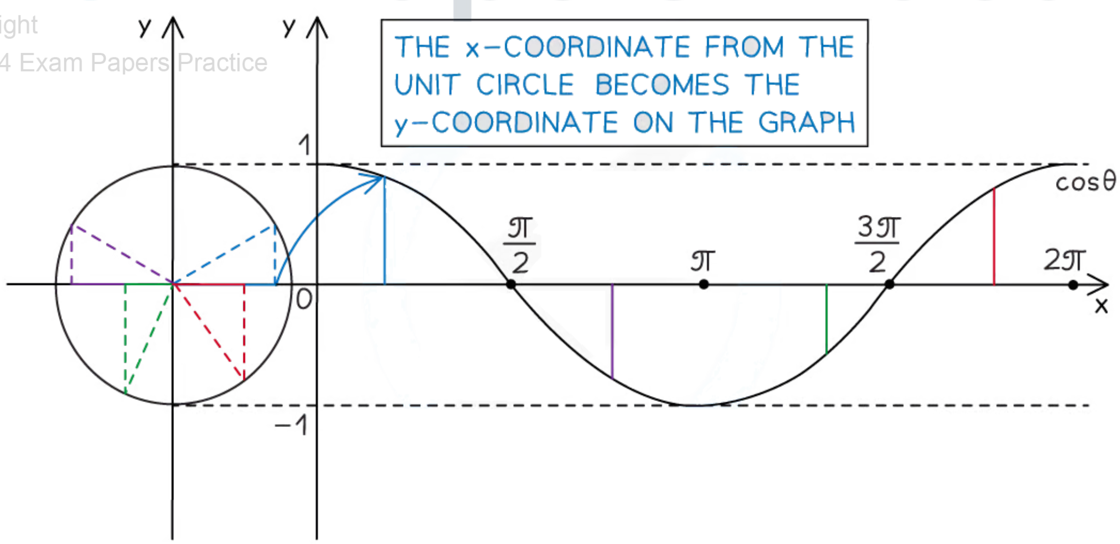
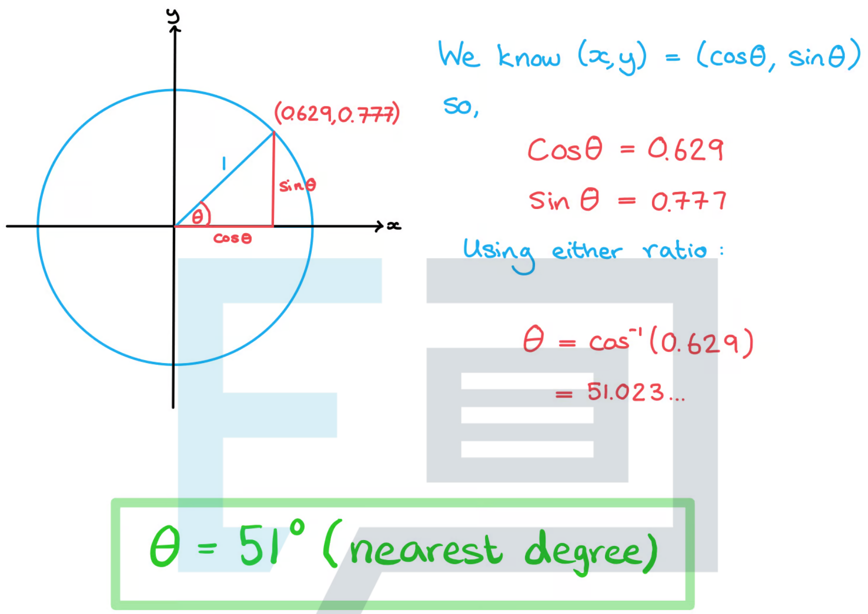


- Looking at the unit circle alongside of the sine or cosine graph will help to visualise this clearer

Worked example

The coordinates of a point on a unit circle, to 3 significant figures, are (0.629, 0.777). Find θ° to the nearest degree.



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Using The Unit Circle

What are the properties of the unit circle?

- The unit circle can be split into four **quadrants** at every 90° ($\frac{\pi}{2}$ rad)
  - The first quadrant is for angles between 0 and 90°
    - **A**ll three of Sinθ, Cosθ and Tanθ are positive in this quadrant
  - The second quadrant is for angles between 90° and 180° ($\frac{\pi}{2}$ rad and π rad)
    - **S**inθ is positive in this quadrant
  - The third quadrant is for angles between 180° and 270° (π rad and $\frac{3\pi}{2}$)
    - **T**anθ is positive in this quadrant
  - The fourth quadrant is for angles between 270° and 360° ($\frac{3\pi}{2}$ rad and 2π)
    - **C**osθ is positive in this quadrant
  - Starting from the **fourth** quadrant (on the bottom right) and working anti-clockwise the positive trig functions spell out **CAST**
    - This is why it is often thought of as the **CAST** diagram
    - You may have your own way of remembering this
    - A popular one starting from the first quadrant is **All Students Take Calculus**
  - To help picture this better try sketching all three trig graphs on one set of axes and look at which graphs are positive in each 90° section

How is the unit circle used to find secondary solutions?

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- Trigonometric functions have more than one input to each output
  - For example sin 30° = sin 150° = 0.5
  - This means that trigonometric equations have more than one solution
  - For example both 30° and 150° satisfy the equation sin x = 0.5
- The unit circle can be used to find all solutions to trigonometric equations in a given interval
  - Your calculator will only give you the first solution to a problem such as $x = \sin^{-1}(0.5)$
    - This solution is called the **primary value**
  - However, due to the **periodic** nature of the trig functions there could be an infinite number of solutions
    - Further solutions are called the **secondary values**
  - This is why you will be given a **domain** in which your solutions should be found
    - This could either be in degrees or in radians
    - If you see π or some multiple of π then you must work in radians
- The following steps may help you use the unit circle to find **secondary values**

STEP 1: Draw the angle into the first quadrant using the x or y coordinate to help you

- If you are working with sin $x = k$, draw the line from the origin to the circumference of the circle at the point where the **y coordinate** is k
- If you are working with cos $x = k$, draw the line from the origin to the circumference of the circle at the point where the **x coordinate** is k
- If you are working with tan $x = k$, draw the line from the origin to the circumference of the circle such that the gradient of the line is k
  - Note that whilst this method works for tan, it is complicated and generally unnecessary, tan x repeats every 180° (π radians) so the quickest method is just to add or subtract multiples of 180° to the primary value
- This will give you the angle which should be measured from the **positive x-axis …**
    - …anticlockwise for a positive angle
    - …clockwise for a negative angle

STEP 2: Draw the radius in the other quadrant which has the same…

- …x-coordinate if solving cos $x = k$
  - This will be the quadrant which is vertical to the original quadrant
- …y-coordinate if solving sin $x = k$
  - This will be the quadrant which is horizontal to the original quadrant
- … gradient if solving tan $x = k$
  - This will be the quadrant diagonally across from the original quadrant

STEP 3: Work out the size of the second angle, measuring from the positive x-axis

- …anticlockwise for a positive angle
- …clockwise for a negative angle
  - You should look at the given range of values to decide whether you need the negative or positive angle

STEP 4: Add or subtract either 360° or 2π radians to both values until you have all solutions in the required range

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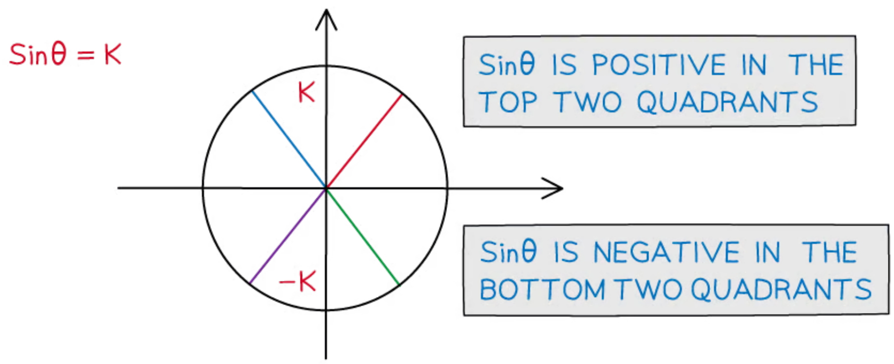

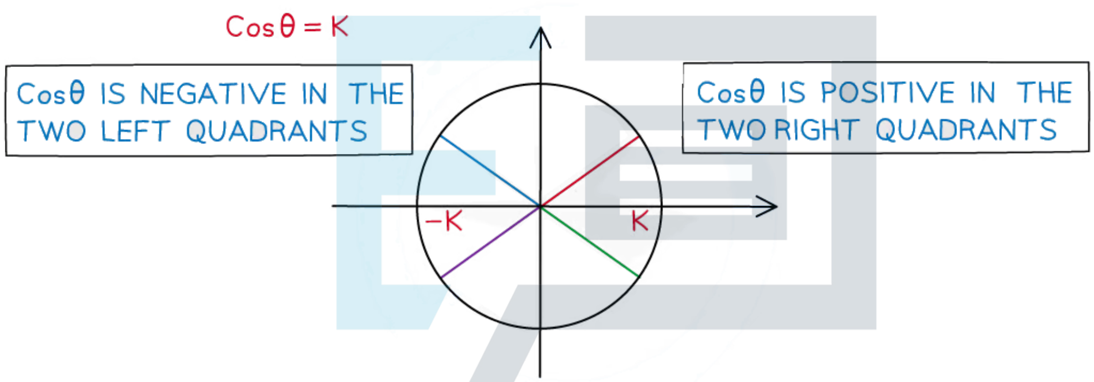

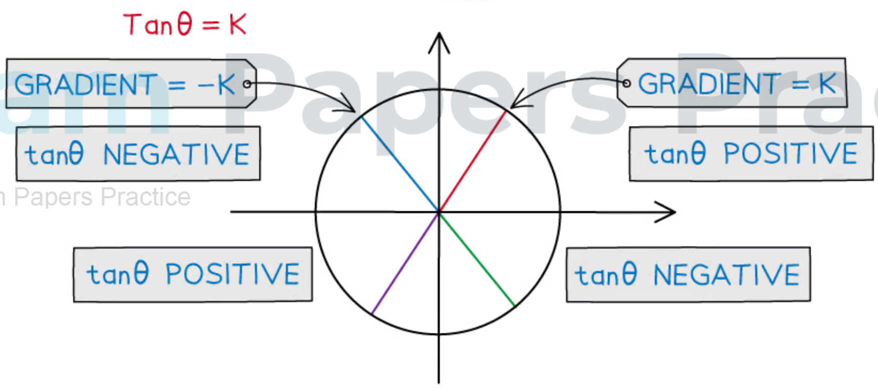

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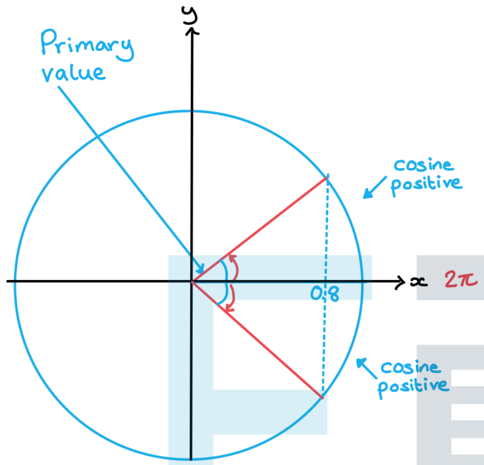
Exam Tip

- Being able to sketch out the unit circle and remembering CAST can help you to find all solutions to a problem in an exam question

Worked example

Given that one solution of $\cos\theta = 0.8$ is $\theta = 0.6435$ radians correct to 4 decimal places, find all other solutions in the range $-2\pi \le \theta \le 2\pi$. Give your answers correct to 3 significant figures.



Cosine is positive in the first and fourth quadrants so draw the angle from the horizontal axis in both quadrants.

Primary value $= 0.6435$

Using diagram, secondary value $= -0.6435$

Therefore all values are: $0.6435 \pm 2\pi n$

and $-0.6435 \pm 2\pi n$

Within given domain: $-2\pi \le \theta \le 2\pi$

$$\theta = -5.64^c,\ -0.644^c,\ 0.644^c,\ 5.64^c$$

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3.4.2 Simple Identities

Simple Identities

What is a trigonometric identity?

- Trigonometric identities are statements that are true for all values of x or θ
- They are used to help simplify trigonometric equations before solving them
- Sometimes you may see identities written with the symbol ≡
  - This means 'identical to'

What trigonometric identities do I need to know?

- The two trigonometric identities you must know are
  - $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
    - This is the identity for $\tan\theta$
  - $\sin^2\theta + \cos^2\theta = 1$
    - This is the Pythagorean identity
    - Note that the notation $\sin^2\theta$ is the same as $(\sin\theta)^2$
- Both identities can be found **in the formula booklet**
- Rearranging the second identity often makes it easier to work with
  - $\sin^2\theta = 1 - \cos^2\theta$
  - $\cos^2\theta = 1 - \sin^2\theta$

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Where do the trigonometric identities come from?

- You do not need to know the proof for these identities but it is a good idea to know where they come from

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- From SOHCAHTOA we know that
  - $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{O}{H}$
  - $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{A}{H}$
  - $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{O}{A}$
- The identity for $\tan\theta$ can be seen by diving $\sin\theta$ by $\cos\theta$?

- $\frac{\sin\theta}{\cos\theta} = \frac{\frac{O}{H}}{\frac{A}{H}} = \frac{O}{A} = \tan\theta$
  - This can also be seen from the unit circle by considering a right-triangle with a hypotenuse of 1
    - $\tan\theta = \frac{O}{A} = \frac{\sin\theta}{\cos\theta}$
- The Pythagorean identity can be seen by considering a right-triangle on the unit circle with a hypotenuse of 1
  - Then $(\text{opposite})^2 + (\text{adjacent})^2 = 1$
  - Therefore $\sin^2\theta + \cos^2\theta = 1$
- Considering the equation of the unit circle also shows the Pythagorean identity
  - The equation of the unit circle is $x^2 + y^2 = 1$
  - The coordinates on the unit circle are $(\cos\theta, \sin\theta)$
  - Therefore the equation of the unit circle could be written $\cos^2\theta + \sin^2\theta = 1$
- A third very useful identity is $\sin\theta = \cos(90° - \theta)$ or $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$
  - This is not included in the formula booklet but is useful to remember

How are the trigonometric identities used?

- Most commonly trigonometric identities are used to change an equation into a form that allows it to be solved
- They can also be used to prove further identities such as the **double angle formulae**

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Exam Tip

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- If you are asked to show that one thing is identical (≡) to another, look at what parts are missing – for example, if tan x has gone it must have been substituted

Worked example

Show that the equation $2\sin^2 x - \cos x = 0$ can be written in the form $a\cos^2 x + b\cos x + c = 0$, where a, b and c are integers to be found.

$2\sin^2 x - \cos x = 0$

Equation has both $\sin x$ and $\cos x$ so will need changing before it can be solved.

Use the identity $\sin^2 x = 1 - \cos^2 x$

Substitute: $2(1 - \cos^2 x) - \cos x = 0$

Expand: $2 - 2\cos^2 x - \cos x = 0$

Rearrange: $2\cos^2 x + \cos x - 2 = 0$

$a = 2$, $b = 1$, $c = -2$

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3.4.3 Solving Trigonometric Equations

Graphs of Trigonometric Functions

What are the graphs of trigonometric functions?

- The trigonometric functions sin, cos and tan all have special **periodic graphs**
- You'll need to know their properties and how to sketch them for a given domain in either **degrees** or **radians**
- Sketching the trigonometric graphs can help to
  - Solve trigonometric equations and find all solutions
  - Understand transformations of trigonometric functions

What are the properties of the graphs of sin x and cos x?

- The graphs of sin x and cos x are both **periodic**
  - They **repeat every 360°** (2π radians)
  - The angle will always be on the x-axis
    - Either in degrees or radians
- The graphs of sin x and cos x are always in the **range** $-1 \le y \le 1$
  - **Domain:** $\{x \mid x \in \mathbb{R}\}$
  - **Range:** $\{y \mid -1 \le y \le 1\}$
  - The graphs of sin x and cos x are identical however one is a **translation** of the other
    - sin x passes through the origin
    - cos x passes through (0, 1)
- The **amplitude** of the graphs of sin x and cos x is 1

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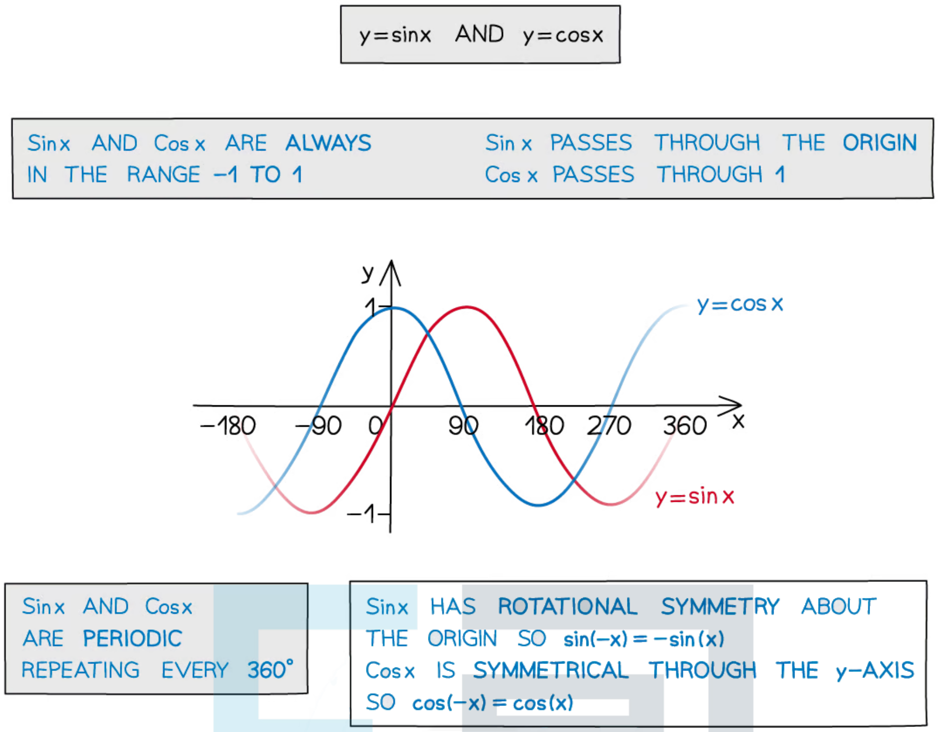
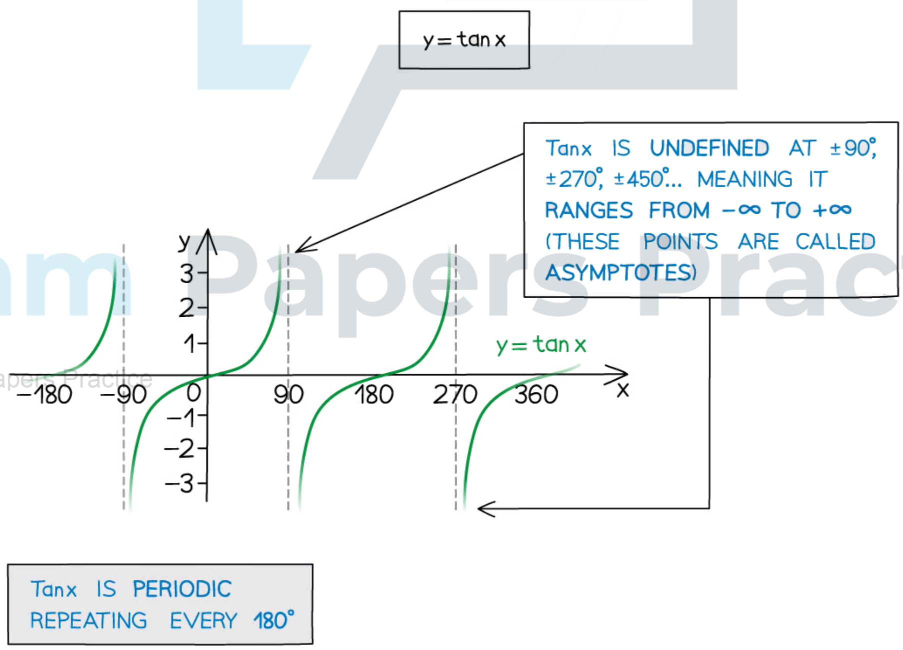
What are the properties of the graph of tan x?

- The graph of tan x is **periodic**
  - It **repeats every 180°** (π radians)
  - The angle will always be on the x-axis
    - Either in degrees or radians
- The graph of tan x is **undefined** at the points ±90°, ±270° etc
  - There are **asymptotes** at these points on the graph
  - In radians this is at the points $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$ etc
- The range of the graph of tan x is
  - **Domain:** $\left\{x \mid x \ne \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\right\}$
  - **Range:** $\{y \mid y \in \mathbb{R}\}$





How do I sketch trigonometric graphs?

- You may need to sketch a trigonometric graph so you will need to remember the key features of each one
- The following steps may help you sketch a trigonometric graph
  - STEP 1: Check whether you should be working in degrees or radians
    - You should check the domain given for this
    - If you see π in the given domain then you should work in radians
  - STEP 2: Label the x-axis in multiples of 90°

   - This will be multiples of $\frac{\pi}{2}$ if you are working in radians
   - Make sure you cover the whole domain on the x-axis
- STEP 3: Label the y-axis
   - The range for the y-axis will be $-1 \leq y \leq 1$ for sin or cos
   - For tan you will not need any specific points on the y-axis
- STEP 4: Draw the graph
   - Knowing exact values will help with this, such as remembering that sin(0) = 0 and cos(0) = 1
   - Mark the important points on the axis first
   - If you are drawing the graph of tan x put the asymptotes in first
   - If you are drawing sin x or cos x mark in where the maximum and minimum points will be
   - Try to keep the symmetry and rotational symmetry as you sketch, as this will help when using the graph to find solutions

Exam Tip

- Sketch all three trig graphs on your exam paper so you can refer to them as many times as you need to!

Worked example

Sketch the graphs of $y = \cos\theta$ and $y = \tan\theta$ on the same set of axes in the interval $-\pi \leq \theta \leq 2\pi$. Clearly mark the key features of both graphs.

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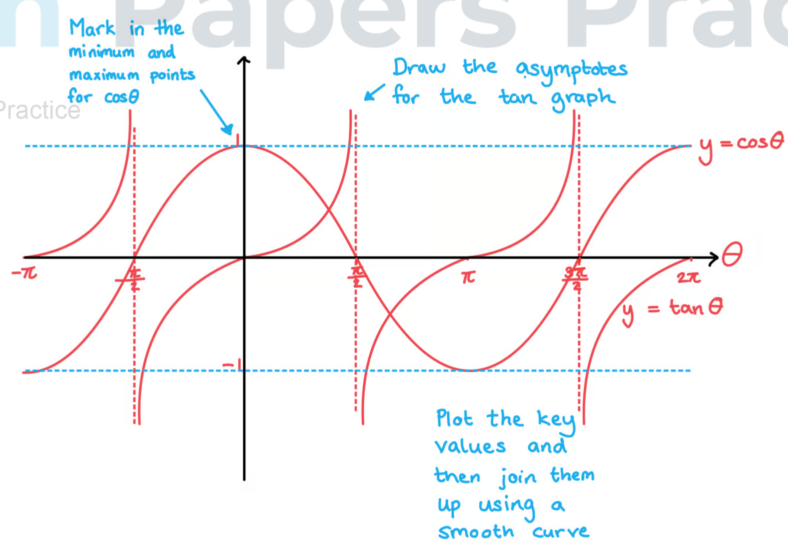

Using Trigonometric Graphs

How can I use a trigonometric graph to find extra solutions?

- Your calculator will only give you the first solution to a problem such as $\sin^{-1}(0.5)$
  - This solution is called the **primary value**
- However, due to the **periodic** nature of the trig functions there could be an infinite number of solutions
  - Further solutions are called the **secondary values**
- This is why you will be given a **domain** (interval) in which your solutions should be found
  - This could either be in degrees or in radians
    - If you see π or some multiple of π then you must work in radians
- The following steps will help you use the **trigonometric graphs** to find **secondary values**
  - STEP 1: Sketch the graph for the given function and interval
    - Check whether you should be working in degrees or radians and label the axes with the key values
  - STEP 2: Draw a horizontal line going through the y-axis at the point you are trying to find the values for
    - For example if you are looking for the solutions to $\sin^{-1}(-0.5)$ then draw the horizontal line going through the y-axis at -0.5
    - The number of times this line cuts the graph is the number of solutions within the given interval
  - STEP 3: Find the primary value and mark it on the graph
    - This will either be an exact value and you should know it
    - Or you will be able to use your calculator to find it
  - STEP 4: Use the symmetry of the graph to find all the solutions in the interval by adding or subtracting from the key values on the graph

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What patterns can be seen from the graphs of trigonometric functions?

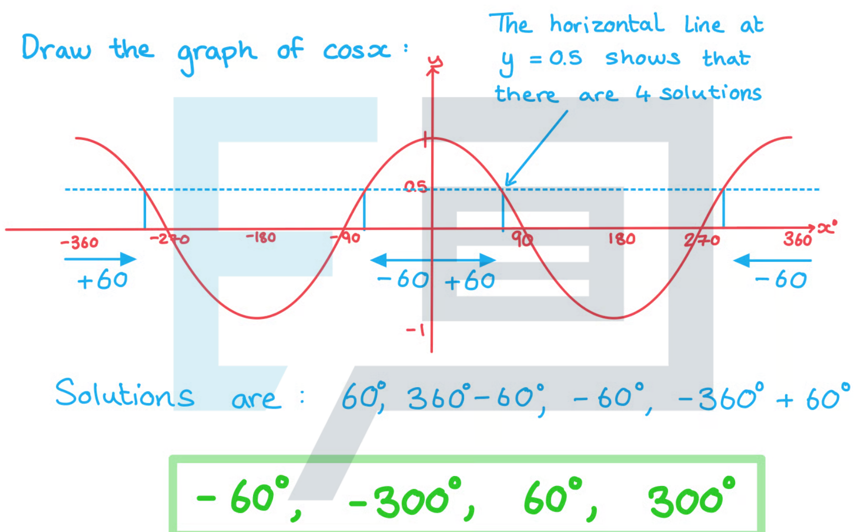
- The graph of sin x has rotational symmetry about the origin
  - So $\sin(-x) = -\sin(x)$
  - $\sin(x) = \sin(180° - x)$ or $\sin(\pi - x)$
- The graph of cos x has reflectional symmetry about the y-axis
  - So $\cos(-x) = \cos(x)$
  - $\cos(x) = \cos(360° - x)$ or $\cos(2\pi - x)$
- The graph of tan x repeats every 180° (π radians)
  - So $\tan(x) = \tan(x \pm 180°)$ or $\tan(x \pm \pi)$
- The graphs of sin x and cos x repeat every 360° (2π radians)
  - So $\sin(x) = \sin(x \pm 360°)$ or $\sin(x \pm 2\pi)$
  - $\cos(x) = \cos(x \pm 360°)$ or $\cos(x \pm 2\pi)$

Exam Tip

- Take care to always check what the **interval** for the angle is that the question is focused on

Worked example

One solution to cos x = 0.5 is 60°. Find all the other solutions in the range -360° ≤ x ≤ 360°.



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