



DP IB Maths: AA HL

3.2 Geometry of 3D Shapes

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3.2.1 3D Coordinate Geometry

3D Coordinate Geometry

How does the 3D coordinate system work?

- In three-dimensional space we can label where any object is using the x-y-z coordinate system
- In the 3D cartesian system, the x- and y- axes usually represent lateral space (length and width) and the z-axis represents vertical height

What can we do with 3D coordinates?

- If we have two points with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) then we should be able to find:
 - The **midpoint** of the two points
 - The **distance** between the two points
- If the coordinates are labelled A and B then the line segment between them is written with the notation [AB]

How do I find the midpoint of two points in 3D?

- The midpoint is the **average (middle) point**
 - It can be found by finding the middle of the x-coordinates and the middle of the y-coordinates
- The coordinates of the midpoint will be

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

- This is given in the formula booklet, you do not need to remember it

How do I find the distance between two points in 3D?

- The distance between two points with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) can be found using the formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

- This is given in the formula booklet, you do not need to remember it

Worked example

The points A and B have coordinates $(-2, 1, 5)$ and $(4, -3, 2)$ respectively.

- i) Calculate the distance of the line segment AB.

Formula for the distance of a line segment:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

↙ in formula booklet

$$A: (-2, 1, 5) \quad B: (4, -3, 2)$$

$\begin{matrix} \uparrow & \uparrow & \uparrow \\ x_1 & y_1 & z_1 \end{matrix} \quad \begin{matrix} \uparrow & \uparrow & \uparrow \\ x_2 & y_2 & z_2 \end{matrix}$

Substitute:

$$\begin{aligned}
 d &= \sqrt{(-2 - 4)^2 + (1 - (-3))^2 + (5 - 2)^2} \\
 &= \sqrt{(-6)^2 + 4^2 + 3^2} \\
 &= \sqrt{36 + 16 + 9} \\
 &= \sqrt{61}
 \end{aligned}$$

$$d = 7.81 \text{ units (3 sf)}$$

- ii) Find the midpoint of [AB].

Formula for the midpoint of a line segment:

$$MP = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

↙ in formula booklet

$$A: (-2, 1, 5) \quad B: (4, -3, 2)$$

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$
 $x_1 \quad y_1 \quad z_1 \quad x_2 \quad y_2 \quad z_2$

Substitute:

$$\begin{aligned} MP &= \left(\frac{-2 + 4}{2}, \frac{1 + (-3)}{2}, \frac{5 + 2}{2} \right) \\ &= \left(\frac{2}{2}, -\frac{2}{2}, \frac{7}{2} \right) \end{aligned}$$

$$MP = (1, -1, 3.5)$$

3.2.2 Volume & Surface Area

Volume of 3D Shapes

What is volume?

- The volume of a 3D shape is a measure of how much 3D space it takes up
 - A 3D shape is also called a **solid**
- You need to be able to calculate the volume of a number of common shapes

How do I find the volume of cuboids, prisms and cylinders?

- A prism is a 3-D shape that has two identical **base** shapes connected by parallel **edges**
 - A prism has the same base shape all the way through
 - A **prism** takes its name from its base
- To find the **volume** of any prism use the formula:

$$\text{Volume of a prism} = Ah$$

- Where **A** is the area of the cross section and **h** is the base height
 - **h** could also be the length of the prism, depending on how it is oriented
- This is in the formula booklet in the **prior learning** section at the beginning
- The base could be any shape so as long as you know its area and length you can calculate the volume of any prism

- Note two special cases:
 - To find the volume of a cuboid use the formula:

$$\text{Volume of a cuboid} = \text{length} \times \text{width} \times \text{height}$$

$$V = lwh$$

- The volume of a **cylinder** can be found in the same way as a prism using the formula:

$$\text{Volume of a cylinder} = \pi r^2 h$$

- where ***r*** is the radius, ***h*** is the height (or length, depending on the orientation)
- Note that a cylinder is technically not a prism as its base is not a polygon, however the method for finding its volume is the same
- Both of these are in the **formula booklet** in the **prior learning** section

How do I find the volume of pyramids and cones?

- In a **right-pyramid** the apex (the joining point of the triangular faces) is vertically above the centre of the base
 - The base can be any shape but is usually a square, rectangle or triangle
- To calculate the volume of a **right-pyramid** use the formula

$$V = \frac{1}{3} Ah$$

- Where ***A*** is the area of the base, ***h*** is the height
- Note that the height must be **vertical to the base**
- A **right cone** is a circular-based pyramid with the vertical height joining the apex to the centre of the circular base
- To calculate the volume of a **right-cone** use the formula

$$V = \frac{1}{3} \pi r^2 h$$

- Where ***r*** is the radius, ***h*** is the height
- These formulae are both given in the formula booklet

How do I find the volume of a sphere?

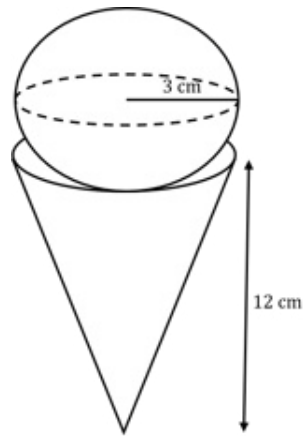
- To calculate the volume of a **sphere** use the formula

$$V = \frac{4}{3} \pi r^3$$

- Where ***r*** is the radius
 - the line segment from the centre of the sphere to the surface
 - This formula is given in the formula booklet

Worked example

A dessert can be modelled as a right-cone of radius 3 cm and height 12 cm and a scoop of ice-cream in the shape of a sphere of radius 3 cm. Find the total volume of the ice-cream and cone.



Volume of a sphere: $V = \frac{4}{3} \pi r^3$ (In formula booklet)

$$\text{Substitute: } r = 3 \Rightarrow V = \frac{4}{3} \pi \times 3^3 \\ = 36\pi$$

Volume of a right cone: $V = \frac{1}{3} \pi r^2 h$ (In formula booklet)

$$\text{Substitute: } r = 3, h = 12 \Rightarrow V = \frac{1}{3} \pi (3)^2 (12) \\ = 36\pi$$

$$\text{Total Volume} = 72\pi \text{ cm}^3$$

$$\text{Total Volume} = 226 \text{ cm}^3 \text{ (3sf)}$$

Surface Area of 3D Shapes

What is surface area?

- The surface area of a 3D shape is the sum of the areas of all the **faces** that make up a shape
 - A **face** is one of the flat or curved surfaces that make up a 3D shape
 - It often helps to consider a 3D shape in the form of its 2D net

How do I find the surface area of cuboids, pyramids and prisms?

- Any prisms and pyramids that have polygons as their bases have only flat faces
 - The surface area is simply found by adding up the areas of these flat faces
 - Drawing a 2D net will help to see which faces the 3D shape is made up of

How do I find the surface area of cylinders, cones and spheres?

- Cones, cylinders and spheres all have curved faces so it is not always as easy to see their shape
 - The net of a **cylinder** is made up of two identical circles and a rectangle
 - The rectangle is the curved surface area and is harder to identify
 - The length of the rectangle is the same as the circumference of the circle
 - The area of the **curved surface area** is

$$A = 2\pi rh$$

- where r is the radius, h is the height
- This is given in the formula book in the prior learning section
- The area of the **total surface area of a cylinder** is

$$A = 2\pi rh + 2\pi r^2$$
 - This is **not** given in the formula book, however it is easy to put together as both the area of a circle and the area of the curved surface area are given
- The net of a **cone** consists of the circular base along with the curved surface area
 - The area of the **curved surface area** is

$$A = \pi rl$$

- Where r is the radius and l is the **slant height**
- This is **given in the formula book**
 - Be careful not to confuse the slant height, l , with the vertical height, h
 - Note that r , h and l will create a **right-angled triangle** with l as the hypotenuse
- The area of the **total surface area of a cone** is

$$A = \pi rl + \pi r^2$$

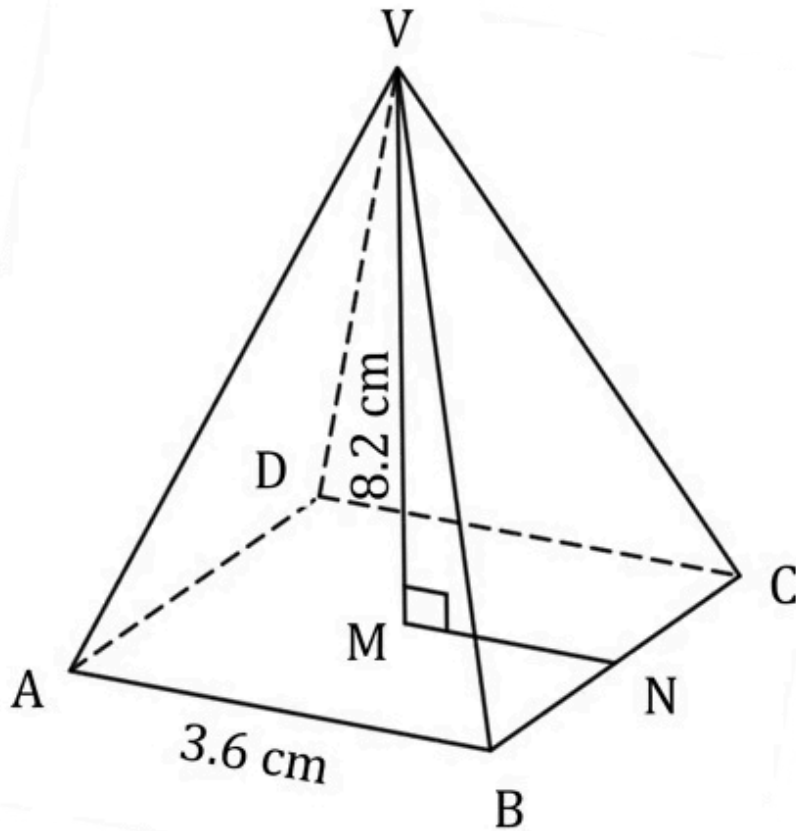
- This is **not** given in the formula book, however it is easy to put together as both the area of a circle and the area of the curved surface area are given
- To find the surface area of a **sphere** use the formula

$$A = 4\pi r^2$$

- where r is the radius (line segment from the centre to the surface)
- This is given in the formula booklet, you do not have to remember it

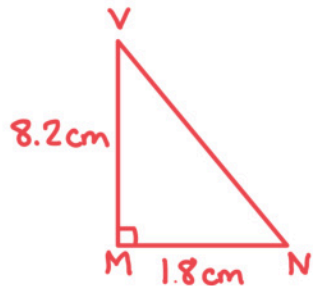
Worked example

In the diagram below $ABCD$ is the square base of a right pyramid with vertex V . The centre of the base is M . The sides of the square base are 3.6 cm and the vertical height is 8.2 cm.



- i) Use the Pythagorean Theorem to find the distance VN .

Sketch the triangle MNV:



M is the midpoint
so $MN = 3.6 \div 2$

By the Pythagorean Theorem:

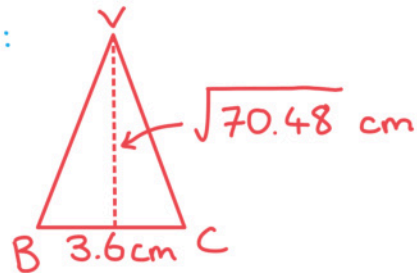
$$\begin{aligned} VN^2 &= \sqrt{VM^2 + MN^2} \\ &= \sqrt{8.2^2 + 1.8^2} \\ &= \sqrt{70.48} \end{aligned}$$

$$VN = 8.40 \text{ cm (3sf)}$$

- ii) Calculate the area of the triangle ABV.

$$\text{Area } \triangle ABV = \text{area } \triangle BCV$$

Sketch $\triangle BCV$:



$$\text{Area of a triangle} = \frac{1}{2}bh$$

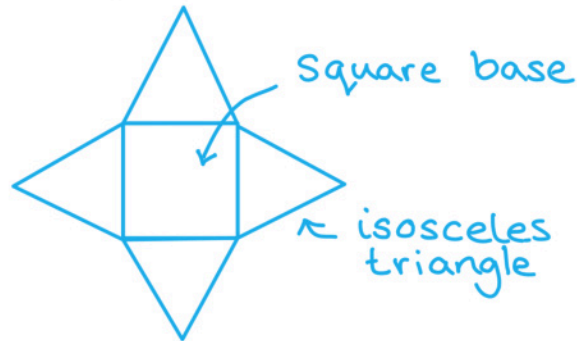
$$\text{Substitute } b = 3.6, h = \sqrt{70.48}$$

$$\begin{aligned}\text{Area} &= \frac{1}{2}(3.6)(\sqrt{70.48}) \\ &= 15.111... \text{ cm}^2\end{aligned}$$

$$\text{Area } \triangle ABV = 15.1 \text{ cm}^2$$

- iii) Find the surface area of the right pyramid.

Considering the net may help:



$$\text{Surface area} = \text{area square} + 4(\text{area triangle})$$

$$\begin{aligned} SA &= 3.6^2 + 4(15.111...) \\ &= 73.405... \text{ cm}^2 \end{aligned}$$

$$SA = 73.4 \text{ cm}^2 \text{ (3sf)}$$