DP IB Maths: AI HL

3.2 Geometry of 3D Shapes

Contents

3.2.1 3D Coordinate Geometry

3D Coordinate Geometry

How does the 3D coordinate system work?

- In three-dimensional space we can label where any object is using the *x*-*y*-*z* coordinate system
- In the 3D cartesian system, the *x*- and *y*- axes usually represent lateral space (length and width) and the *z*-axis represents vertical height

What can we do with 3D coordinates?

- If we have two points with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) then we should be able to find:
  - The **midpoint** of the two points
  - The **distance** between the two points
- If the coordinates are labelled A and B then the line segment between them is written with the notation [AB]

How do I find the midpoint of two points in 3D?

- The midpoint is the **average (middle) point**
  - It can be found by finding the middle of the *x*-coordinates and the middle of the *y*-coordinates
- The coordinates of the midpoint will be

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

  - This is given in the formula booklet, you do not need to remember it

How do I find the distance between two points in 3D?

- The distance between two points with coordinates $((x_1, y_1, z_1)$ and (x_2, y_2, z_2) can be found using the formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

  - This is given in the formula booklet, you do not need to remember it

Worked example

The points *A* and *B* have coordinates (-2, 1, 5) and (4, -3, 2) respectively.

i) Calculate the distance of the line segment *AB*.

Formula for the distance of a line segment:

in formula booklet

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

$$A : (\underset{x_1}{-2}, \underset{y_1}{1}, \underset{z_1}{5}) \qquad B : (\underset{x_2}{4}, \underset{y_2}{-3}, \underset{z_2}{2})$$

Substitute:

$$d = \sqrt{(-2-4)^2 + (1-(-3))^2 + (5-2)^2}$$

$$= \sqrt{(-6)^2 + 4^2 + 3^2}$$

$$= \sqrt{36 + 16 + 9}$$

$$= \sqrt{61}$$

$$\boxed{d = 7.81 \text{ units } (3 \text{ sf})}$$

ii) Find the midpoint of [*AB*].

Formula for the midpoint of a line segment:

$$MP = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

↙ in formula booklet

$$A : (\underset{\overset{\uparrow}{x_1}}{-2}, \underset{\overset{\uparrow}{y_1}}{1}, \underset{\overset{\uparrow}{z_1}}{5}) \qquad B : (\underset{\overset{\uparrow}{x_2}}{4}, \underset{\overset{\uparrow}{y_2}}{-3}, \underset{\overset{\uparrow}{z_2}}{2})$$

Substitute:

$$MP = \left(\frac{-2+4}{2}, \frac{1+(-3)}{2}, \frac{5+2}{2}\right)$$

$$= \left(\frac{2}{2}, -\frac{2}{2}, \frac{7}{2}\right)$$

$$\boxed{MP = (1, -1, 3.5)}$$

3.2.2 Volume & Surface Area

Volume of 3D Shapes

What is volume?

- The volume of a 3D shape is a measure of how much 3D space it takes up
    - A 3D shape is also called a **solid**
- You need to be able to calculate the volume of a number of common shapes

How do I find the volume of cuboids, prisms and cylinders?

- A prism is a 3–D shape that has two identical **base** shapes connected by parallel **edges**
    - A prism has the same base shape all the way through
    - A **prism** takes its name from its base
- To find the **volume** of any prism use the formula:

$$\text{Volume of a prism} = Ah$$

    - Where ***A*** is the area of the cross section and ***h*** is the base height
        - *h* could also be the length of the prism, depending on how it is oriented
    - This is in the formula booklet in the **prior learning** section at the beginning
    - The base could be any shape so as long as you know its area and length you can calculate the volume of any prism

- Note two special cases:
  - To find the volume of a cuboid use the formula:

$$\text{Volume of a cuboid} = \text{length} \times \text{width} \times \text{height}$$

$$V = lwh$$

  - The volume of a **cylinder** can be found in the same way as a prism using the formula:

$$\text{Volume of a cylinder} = \pi r^2 h$$

  - where r is the radius, h is the height (or length, depending on the orientation
  - Note that a cylinder is technically not a prism as its base is not a polygon, however the method for finding its volume is the same
- Both of these are **in the formula booklet** in the **prior learning** section

How do I find the volume of pyramids and cones?

- In a **right-pyramid** the apex (the joining point of the triangular faces) is vertically above the centre of the base
  - The base can be any shape but is usually a square, rectangle or triangle
- To calculate the volume of a **right-pyramid** use the formula

$$V = \frac{1}{3} Ah$$

  - Where A is the area of the base, h is the height
  - Note that the height must be **vertical to the base**
- **A right cone** is a circular-based pyramid with the vertical height joining the apex to the centre of the circular base
- To calculate the volume of a **right-cone** use the formula

$$V = \frac{1}{3} \pi r^2 h$$

  - Where r is the radius, h is the height
- These formulae are both given in the formula booklet

How do I find the volume of a sphere?

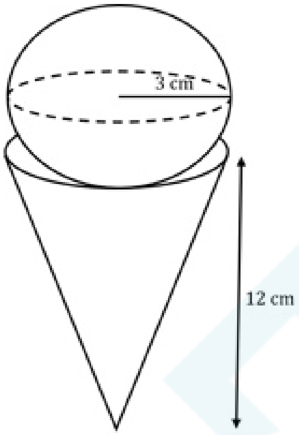
- To calculate the volume of a **sphere** use the formula

$$V = \frac{4}{3} \pi r^3$$

  - Where r is the radius
    - the line segment from the centre of the sphere to the surface
  - This formula is given in the formula booklet

Worked example

A dessert can be modelled as a right-cone of radius 3 cm and height 12 cm and a scoop of ice-cream in the shape of a sphere of radius 3 cm. Find the total volume of the ice-cream and cone.



Volume of a sphere: $V = \frac{4}{3}\pi r^3$ (In formula booklet)

Substitute: $r = 3 \Rightarrow V = \frac{4}{3}\pi \times 3^3$

$= 36\pi$

Volume of a right cone: $V = \frac{1}{3}\pi r^2 h$ (In formula booklet)

Substitute: $r = 3, \quad h = 12 \Rightarrow V = \frac{1}{3}\pi(3)^2(12)$

$= 36\pi$

Total Volume $= 72\pi \text{ cm}^3$

$\boxed{\text{Total Volume} = 226\text{cm}^3 \text{ (3sf)}}$

Surface Area of 3D Shapes

What is surface area?

- The surface area of a 3D shape is the sum of the areas of all the **faces** that make up a shape
  - A **face** is one of the flat or curved surfaces that make up a 3D shape
  - It often helps to consider a 3D shape in the form of its 2D net

How do I find the surface area of cuboids, pyramids and prisms?

- Any prisms and pyramids that have polygons as their bases have only flat faces
  - The surface area is simply found by adding up the areas of these flat faces
  - Drawing a 2D net will help to see which faces the 3D shape is made up of

How do I find the surface area of cylinders, cones and spheres?

- Cones, cylinders and spheres all have curved faces so it is not always as easy to see their shape
  - The net of a **cylinder** is made up of two identical circles and a rectangle
  - The rectangle is the curved surface area and is harder to identify
  - The length of the rectangle is the same as the circumference of the circle
  - The area of the **curved surface area** is

$$A = 2\pi rh$$

    - where r is the radius, h is the height
  - This is given in the formula book in the prior learning section
  - The area of the **total surface area of a cylinder** is

$$A = 2\pi rh + 2\pi r^2$$

  - This is **not** given in the formula book, however it is easy to put together as both the area of a circle and the area of the curved surface area are given
- The net of a **cone** consists of the circular base along with the curved surface area
  - The area of the **curved surface area** is

$$A = \pi rl$$

    - Where r is the radius and l is the **slant height**
  - This is **given in the formula book**
    - Be careful not to confuse the slant height, l, with the vertical height, h
    - Note that r, h and l will create a **right-triangle** with l as the hypotenuse
  - The area of the **total surface area of a cone** is

$$A = \pi rl + \pi r^2$$

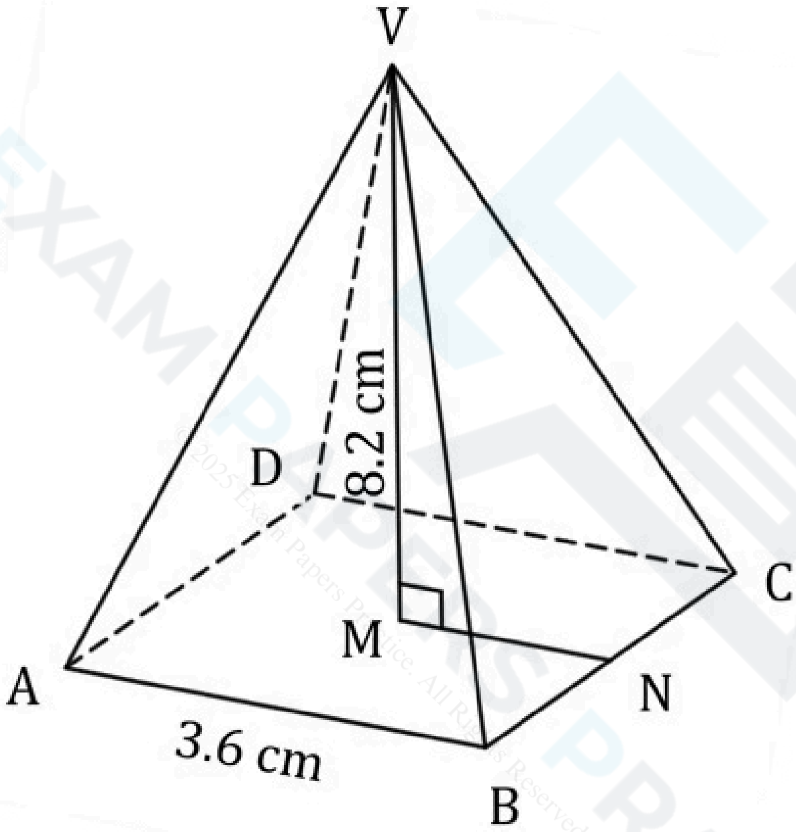
  - This is **not** given in the formula book, however it is easy to put together as both the area of a circle and the area of the curved surface area are given
- To find the surface area of a **sphere** use the formula

$$A = 4\pi r^2$$

  - where r is the radius (line segment from the centre to the surface)
- This is given in the formula booklet, you do not have to remember it

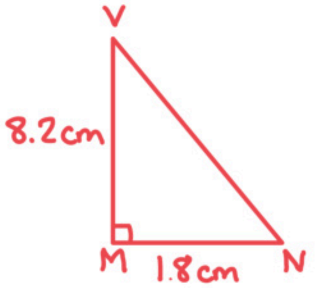
Worked example

In the diagram below *ABCD* is the square base of a right pyramid with vertex *V*. The centre of the base is *M*. The sides of the square base are 3.6 cm and the vertical height is 8.2 cm.



i) Use the Pythagorean Theorem to find the distance VN.

Sketch the triangle MNV:



M is the midpoint
so MN = 3.6 ÷ 2

By the Pythagorean Theorem:

$$VN^2 = \sqrt{VM^2 + MN^2}$$

$$= \sqrt{8.2^2 + 1.8^2}$$

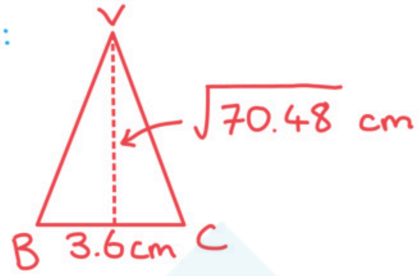
$$= \sqrt{70.48}$$

$$\boxed{VN = 8.40 \text{ cm } (3sf)}$$

ii) Calculate the area of the triangle ABV.

Area ΔABV = area ΔBCV

Sketch ΔBCV:



Area of a triangle $= \frac{1}{2}bh$

Substitute $b = 3.6$, $h = \sqrt{70.48}$

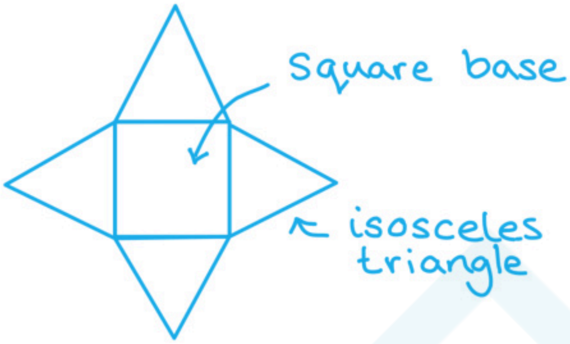
$$\text{Area} = \frac{1}{2}(3.6)(\sqrt{70.48})$$

$$= 15.111\ldots \text{ cm}^2$$

$$\boxed{\text{Area } \Delta ABV = 15.1 \text{ cm}^2}$$

iii) Find the surface area of the right pyramid.

Considering the net may help:



Surface area = area square + 4(area triangle)

$SA = 3.6^2 + 4(15.111...)$

$= 73.405... \text{ cm}^2$

$\boxed{SA = 73.4 \text{ cm}^2 \text{ (3sf)}}$