3.11 Vector Planes

IB Maths - Revision Notes

AA HL

3.11.1 Vector Equations of Planes

Equation of a Plane in Vector Form

How do I find the vector equation of a plane?

- A plane is a flat surface which is two-dimensional
  - Imagine a flat piece of paper that continues on forever in both directions
- A plane in often denoted using the capital Greek letter Π
- The vector form of the equation of a plane can be found using **two direction vectors** on the plane
  - The direction vectors must be
    - **parallel** to the plane
    - **not parallel** to each other
    - therefore they will **intersect** at some point on the plane
- The formula for finding the **vector equation** of a plane is
  - $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$
    - Where $\boldsymbol{r}$ is the **position vector** of any point on the plane
    - $\boldsymbol{a}$ is the **position vector** of a known point on the plane
    - $\boldsymbol{b}$ and $\boldsymbol{c}$ are two **non-parallel direction** (displacement) **vectors** parallel to the plane
    - λ and μ are scalars
  - The formula is **given in the formula booklet** but you must make sure you know what each part means
- As $\boldsymbol{a}$ could be the position vector of **any** point on the plane and $\boldsymbol{b}$ and $\boldsymbol{c}$ could be **any non-parallel** direction vectors on the plane there are infinite vector equations for a single plane

How do I determine whether a point lies on a plane?

- Given the equation of a plane $\boldsymbol{r} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix} + \lambda\begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{pmatrix} + \mu\begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \boldsymbol{c}_3 \end{pmatrix}$ then the point $\boldsymbol{r}$ with position vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is on the plane if there exists a value of λ and μ such that
  - $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix} + \lambda\begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{pmatrix} + \mu\begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \boldsymbol{c}_3 \end{pmatrix}$

- This means that there exists a single value of λ and μ that satisfy the three **parametric** equations:
  - $x = a_1 + \lambda b_1 + \mu c_1$
  - $y = a_2 + \lambda b_2 + \mu c_2$
  - $z = a_3 + \lambda b_3 + \mu c_3$
- Solve two of the equations first to find the values of λ and μ that satisfy the first two equation and then check that this value also satisfies the third equation
- If the values of λ and μ do not satisfy all three equations, then the point $\boldsymbol{r}$ does not lie on the plane

Exam Tip

- The formula for the vector equation of a plane is given in the formula booklet, make sure you know what each part means
- Be careful to use different letters, e.g. λ and μ as the scalar multiples of the two direction vectors

Worked example

The points A, B and C have position vectors $\boldsymbol{a} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\boldsymbol{b} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, and $\boldsymbol{c} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ respectively, relative to the origin O.

(a) Find the vector equation of the plane.

Start by finding the direction vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix}$$

$$\overrightarrow{AC} = \underline{c} - \underline{a} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$$

All three points lie on the plane, so choose the position vector of one point, e.g. $\overrightarrow{OA}$, to use as 'a' in the vector equation of a plane formula.

Check that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not parallel.

$$r = \underline{a} + \lambda \overrightarrow{AB} + \mu \overrightarrow{AC}$$

$$\boxed{\underline{r} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}}$$

(This is one of many correct answers)

(b) Determine whether the point D with coordinates (-2, -3, 5) lies on the plane.

Let D have position vector $\underline{d} = \begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix}$, then the point D lies on the plane if there exists a value of λ and μ for which: $\begin{pmatrix} -2 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$

Find the parametric equations:

$-2 = 3 - 2\lambda + \mu \Rightarrow \mu - 2\lambda = -5$ ① solve two equations

$-3 = 2 - 4\lambda - 3\mu \Rightarrow 3\mu + 4\lambda = 5$ ② for λ and μ.

$5 = -1 + 5\lambda + 4\mu \Rightarrow 4\mu + 5\lambda = 6$ ③

Find the value of λ and μ from two equations:

2①: $2\mu - 4\lambda = -10$

+ ②: $3\mu + 4\lambda = 5$

$5\mu = -5$

$\mu = -1$ sub into ①: $(-1) - 2\lambda = -5$

$\lambda = 2$

Check to see if λ and μ satisfy the third equation:

$4(-1) + 5(2) = -4 + 10 = 6$ ✓

$\boxed{\text{The point D lies on the plane.}}$

Equation of a Plane in Cartesian Form

How do I find the vector equation of a plane in cartesian form?

- The **cartesian** equation of a plane is given in the form
  - $ax + by + cz = d$
  - This is **given in the formula booklet**
- A **normal vector** to the plane can be used along with a **known point on the plane** to find the cartesian equation of the plane
  - The normal vector will be a vector that is **perpendicular** to the plane
- The **scalar product** of the normal vector and any **direction vector** on the plane will the **zero**
  - The two vectors will be perpendicular to each other
  - The **direction vector** from a fixed-point A to any point on the plane, R can be written as $\boldsymbol{r} - \boldsymbol{a}$
  - Then $\boldsymbol{n} \cdot (\boldsymbol{r} - \boldsymbol{a}) = 0$ and it follows that $(\boldsymbol{n} \cdot \boldsymbol{r}) - (\boldsymbol{n} \cdot \boldsymbol{a}) = 0$
- This gives the **equation of a plane using the normal vector:**
  - $\boldsymbol{n} \cdot \boldsymbol{r} = \boldsymbol{a} \cdot \boldsymbol{n}$
    - Where $\boldsymbol{r}$ is the **position vector** of any point on the plane
    - $\boldsymbol{a}$ is the **position vector** of a known point on the plane
    - $\boldsymbol{n}$ is a vector that is **normal** to the plane
  - This is **given in the formula booklet**
- If the vector $\boldsymbol{r}$ is given in the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\boldsymbol{a}$ and $\boldsymbol{n}$ are both known vectors given in the form $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ then the Cartesian equation of the plane can be found using:

  - $\boldsymbol{n} \cdot \boldsymbol{r} = n_1 x + n_2 y + n_3 z$
  - $\boldsymbol{a} \cdot \boldsymbol{n} = a_1 n_1 + a_2 n_2 + a_3 n_3$
  - Therefore $n_1 x + n_2 y + n_3 z = a_1 n_1 + a_2 n_2 + a_3 n_3$
  - This simplifies to the form $ax + by + cz = d$

How do I find the equation of a plane in Cartesian form given the vector form?

- The **Cartesian** equation of a plane can be found if you know
  - the **normal vector** and
  - a **point** on the plane

- The **vector equation of a plane** can be used to find the **normal vector** by finding the **vector product** of the two direction vectors
  - A vector product is always perpendicular to the two vectors from which it was calculated
- The vector $\boldsymbol{a}$ given in the vector equation of a plane is a **known point** on the plane
  - Once you have found the normal vector then the point $\boldsymbol{a}$ can be used in the formula $\boldsymbol{n} \cdot \boldsymbol{r} = \boldsymbol{a} \cdot \boldsymbol{n}$ to find the equation in Cartesian form
- To find $ax + by + cz = d$ given $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$:
  - Let $\boldsymbol{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \boldsymbol{b} \times \boldsymbol{c}$ then $d = \boldsymbol{n} \cdot \boldsymbol{a}$

Exam Tip

- In an exam, using whichever form of the equation of the plane to write down a normal vector to the plane is always a good starting point

Worked example

A plane Π contains the point A$(2, 6, -3)$ and has a normal vector $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$.

a) Find the equation of the plane in its Cartesian form.

The components of the normal vector are the x-, y- and z-coefficients of the Cartesian form:

$3x - y + 4z = d$

The point $(2, 6, -3)$ is on the plane so

$d = 3(2) - (6) + 4(-3) = 6 - 6 - 12 = -12$

Therefore $\boxed{3x - y + 4z = -12}$

b) Determine whether point B with coordinates $(-1, 0, -2)$ lies on the same plane.

Test by putting the coordinates into the equation:

$3(-1) - (0) + 4(-2) = -3 - 8 = -11 \neq -12$

The point with coordinates $(-1, 0, 2)$ does not lie on the plane

3.11.2 Intersections of Lines & Planes

Intersection of Line & Plane

How do I tell if a line is parallel to a plane?

- A line is parallel to a plane if its **direction vector** is **perpendicular** to the plane's **normal vector**
- If you know the Cartesian equation of the plane in the form $ax + by + cz = d$ then the values of *a, b,* and *c* are the individual components of a normal vector to the plane
- The **scalar product** can be used to check in the direction vector and the normal vector are perpendicular
  - If two vectors are perpendicular their scalar product will be zero

How do I tell if the line lies inside the plane?

- If the line is parallel to the plane then it will either **never intersect** or it will lie inside the plane
  - Check to see if they have a common point
- If a line is parallel to a plane and they share **any point**, then the line lies inside the plane

How do I find the point of intersection of a line and a plane?

- If a line is **not parallel** to a plane it will **intersect** it at a single point
- If both the **vector equation of the line** and the **Cartesian equation of the plane** is known then this can be found by:
- STEP 1: Set the position vector of the point you are looking for to have the individual components *x, y,* and *z* and substitute into the vector equation of the line
  - $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$
- STEP 2: Find the parametric equations in terms of *x, y,* and *z*
  - $x = x_0 + \lambda l$
  - $y = y_0 + \lambda m$
  - $z = z_0 + \lambda n$
- STEP 3: Substitute these parametric equations into the Cartesian equation of the plane and solve to find λ
  - $a(x_0 + \lambda l) + b(y_0 + \lambda m) + c(z_0 + \lambda n) = d$

- STEP 4: Substitute this value of λ back into the vector equation of the line and use it to find the position vector of the point of intersection
- STEP 5: Check this value in the Cartesian equation of the plane to make sure you have the correct answer

Worked example

Find the point of intersection of the line $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ with the plane

$3x - 4y + z = 8.$

Find the parametric form of the equation of the line:

Let $r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ then $x = 1 + 2\lambda$, $y = -3 - \lambda$, $z = 2 - \lambda$

Substitute into the equation of the plane:

$3(1 + 2\lambda) - 4(-3 - \lambda) + (2 - \lambda) = 8$

Solve to find λ:

$3 + 6\lambda + 12 + 4\lambda + 2 - \lambda = 8$

$\lambda = -1$

Substitute $\lambda = -1$ into the vector equation of the line:

$r = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + (-1)\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 - 2 \\ -3 + 1 \\ 2 + 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$

$\boxed{(-1, -2, 3)}$

Intersection of Planes

How do we find the line of intersection of two planes?

- Two planes will either be **parallel** or they will intersect along a **line**
  - Consider the point where a wall meets a floor or a ceiling
  - You will need to find the **equation of the line** of intersection
- If you have the Cartesian forms of the two planes then the equation of the line of intersection can be found by solving the two equations simultaneously
  - As the solution is a vector equation of a line rather than a unique point you will see below how the equation of the line can be found by part solving the equations
  - For example:
    - $2x - y + 3z = 7$ (1)
    - $x - 3y + 4z = 11$ (2)
- STEP 1: Choose one variable and substitute this variable for λ in both equations
  - For example, letting $x = \lambda$ gives:
    - $2\lambda - y + 3z = 7$ (1)
    - $\lambda - 3y + 4z = 11$ (2)
- STEP 2: Rearrange the two equations to bring λ to one side
  - Equations (1) and (2) become
    - $y - 3z = 2\lambda - 7$ (1)
    - $3y - 4z = \lambda - 11$ (2)
- STEP 3: Solve the equations simultaneously to find the two variables in terms of λ
  - 3(1) – (2) Gives
    - $z = 2 - \lambda$
  - Substituting this into (1) gives
    - $y = -1 - \lambda$

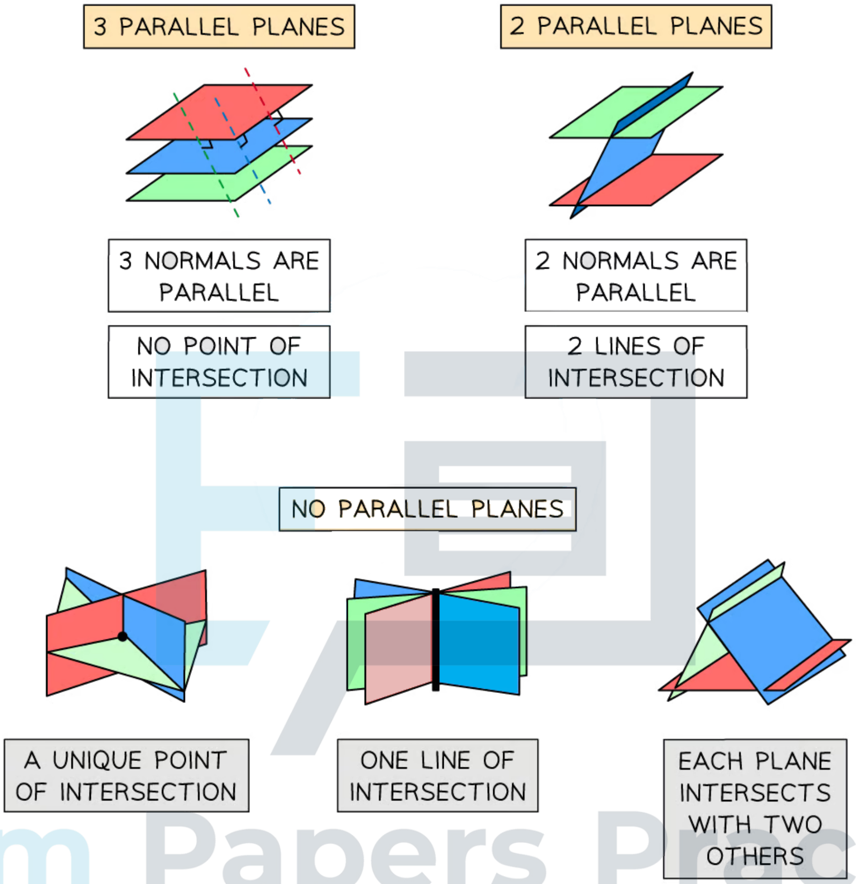
- STEP 4: Write the three parametric equations for x, y, and z in terms of λ and convert into the vector equation of a line in the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$
  - The parametric equations
    - $x = \lambda$
    - $y = -1 - \lambda$
    - $z = 2 - \lambda$
  - Become

- $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$
- If you have fractions in your direction vector you can change its magnitude by multiplying each one by their common denominator
  - The magnitude of the direction vector can be changed without changing the equation of a line
- An alternative method is to find two points on both planes by setting either *x*, *y*, or *z* to zero and solving the system of equations using your GDC or row reduction
  - Repeat this twice to get two points on both planes
  - These two points can then be used to find the vector equation of the line between them
  - This will be the line of intersection of the planes
  - This method relies on the line of intersection having points where the chosen variables are equal to zero

How do we find the relationship between three planes?

- Three planes could either be **parallel**, intersect at one **point**, or intersect along a **line**
- If the three planes have a **unique point of intersection** this point can be found by using your GDC (or row reduction) to solve the three equations in their Cartesian form
  - Make sure you know how to use your GDC to solve a **system of linear equations**
  - Enter all three equations in for the three variables *x*, *y*, and *z*
  - Your GDC will give you the unique solution which will be the coordinates of the point of intersection
- If the three planes do not intersect at a unique point you will not be able to use your GDC to solve the equations
  - If there are no solutions to the system of Cartesian equations then there is no unique point of intersection
- If the three planes are all **parallel** their **normal vectors** will be parallel to each other
  - Show that the normal vectors all have equivalent **direction vectors**
  - These direction vectors may be **scalar multiples** of each other
- If the three planes have **no point of intersection** and are **not all parallel** they may have a relationship such as:
  - Each plane intersects two other planes such that they form a **prism** (none are parallel)
  - Two planes are parallel with the third plane intersecting each of them
  - Check the normal vectors to see if any two of the planes are parallel to decide which relationship they have
- If the three planes intersect along a line there will not be a unique solution to the three equations but there will be a **vector equation of a line** that will satisfy the three equations
- The system of equations will need to be solved by **elimination** or **row reduction**
  - Choose one variable to substitute for λ
  - Solve two of the equations simultaneously to find the other two variables in terms of λ

- Write $x, y,$ and z in terms of λ in the parametric form of the equation of the line and convert into the vector form of the equation of a line



Exam Tip

- In an exam you may need to decide the relationship between three planes by using row reduction to determine the number of solutions
  - Make sure you are confident using row reduction to solve systems of linear equations
  - Make sure you remember the different forms three planes can take

Worked example

Two planes Π_1 and Π_2 are defined by the equations:

Π_1: $3x + 4y + 2z = 7$

Π_2: $x - 2y + 3z = 5$

Find the vector equation of the line of intersection of the two planes.

STEP 1: Let $z = \lambda$, then $3x + 4y + 2\lambda = 7$ ①

$x - 2y + 3\lambda = 5$ ②

You can substitute any variable here, look at the equations to see which is easiest.

STEP 2: ① : $3x + 4y = 7 - 2\lambda$

② : $x - 2y = 5 - 3\lambda$

Write the two equations as simultaneous equations for the two remaining constants.

STEP 3: Find x and y in terms of λ:

① − 2②: $(3x + 4y = 7 - 2\lambda)$

$+ (2x - 4y = 10 - 6\lambda)$

$5x = 17 - 8\lambda$

$x = \frac{17}{5} - \frac{8\lambda}{5}$

sub into ② $\frac{17}{5} - \frac{8\lambda}{5} - 2y + 3\lambda = 5$

$y = \frac{7\lambda}{10} - \frac{8}{10}$

STEP 4: $x = \frac{17}{5} - \frac{8\lambda}{5}$

$y = \frac{7\lambda}{10} - \frac{4}{5}$

$z = \lambda$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{17}{5} \\ -\frac{4}{5} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -\frac{8}{5} \\ \frac{7}{10} \\ 1 \end{pmatrix}$$

The components of the direction vector can be multiplied by a scalar without changing the direction.

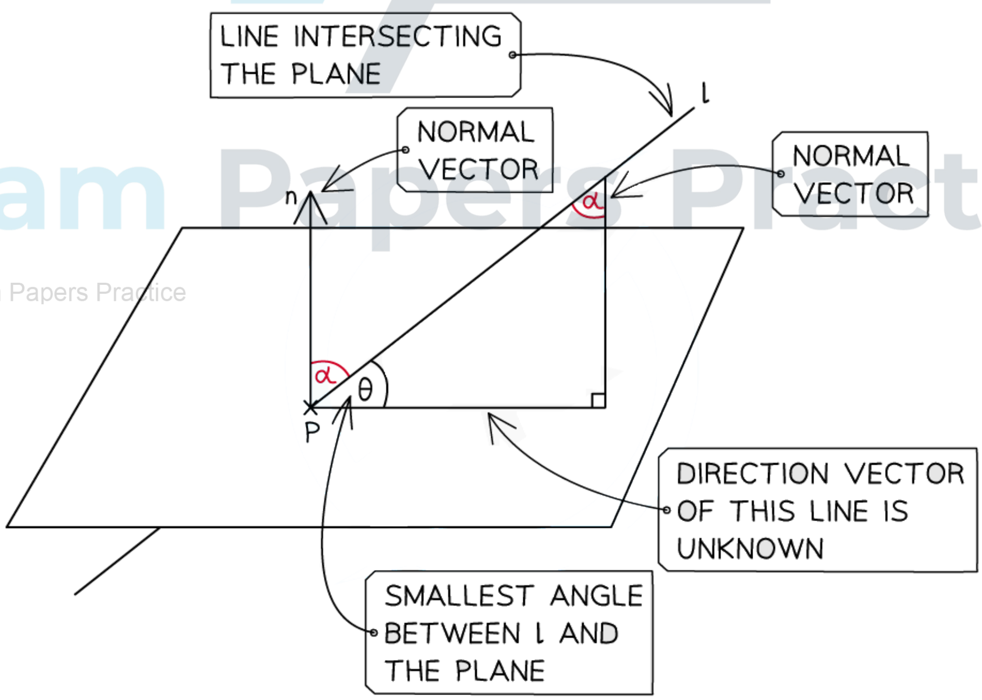
$$\boxed{\mathbf{r} = \begin{pmatrix} 17/5 \\ -4/5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 16 \\ 7 \\ 10 \end{pmatrix}}$$

3.11.3 Angles Between Lines & Planes

Angle Between Line & Plane

How do I find the angle between a line and a plane?

- When you find the angle between a line and a plane you will be finding the angle between the line itself and the line on the plane that creates the smallest angle with it
  - This means the line on the plane directly under the line as it joins the plane
- It is easiest to think of these two lines making a right-triangle with the normal vector to the plane
  - The line joining the plane will be the **hypotenuse**
  - The line on the plane will be **adjacent** to the angle
  - The normal will the **opposite** the angle
- As you do not know the angle of the line on the plane you can instead find the angle between the **normal** and the **hypotenuse**
  - This is the angle **opposite** the angle you want to find
  - This angle **can be found** because you will know the direction vector of the line joining the plane and the normal vector to the plane
  - This angle is also equal to the angle made by the line at the point it joins the plane and the normal vector at this point
- The smallest angle between the line and the plane will be 90° minus the angle between the normal vector and the line
  - In radians this will be $\frac{\pi}{2}$ minus the angle between the normal vector and the line



Exam Tip

- Remember that if the scalar product is negative your answer will result in an obtuse angle
  - Taking the absolute value of the scalar product will ensure that you get the acute angle as your answer

Worked example

Find the angle in radians between the line L with vector equation $\boldsymbol{r} = (2-\lambda)\boldsymbol{i} + (\lambda+1)\boldsymbol{j} + (1-2\lambda)\boldsymbol{k}$ and the plane Π with Cartesian equation $x - 3y + 2z = 5$.

Rewrite line equation in standard vector form:

$$r = \begin{pmatrix} 2-\lambda \\ 1+\lambda \\ 1-2\lambda \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

direction vector of the line

Find the normal vector of the plane:

$$\underbrace{x - 3y + 2z}_{\text{components of the normal vector}} = 5 \implies \text{normal vector} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

Find the angle between the direction vector and the normal vector, α:

Angle between two vectors	$\cos\theta = \dfrac{v_1w_1 + v_2w_2 + v_3w_3}{\lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert}$

$$\cos\alpha = \frac{\left| \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \right|}{\sqrt{(-1)^2 + (1)^2 + (-2)^2} \times \sqrt{1^2 + (-3)^2 + 2^2}} = \frac{|(-1)(1) + (1)(-3) + (-2)(2)|}{\sqrt{6}\sqrt{14}}$$

$$\theta = \frac{\pi}{2} - \cos^{-1}\alpha$$

$$\theta = \frac{\pi}{2} - \cos^{-1}\left(\frac{|-8|}{\sqrt{6}\sqrt{14}}\right)$$

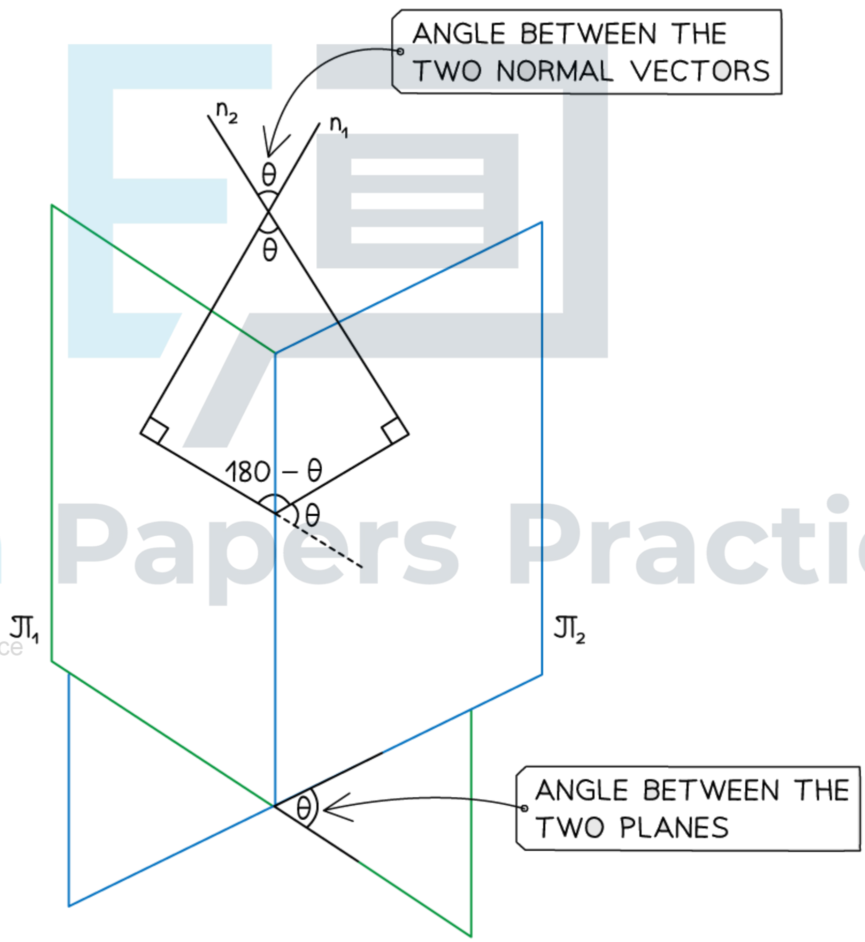
Using the absolute value ensures we find the acute angle.

$$\boxed{\theta = 1.06 \text{ radians (3 s.f.)}}$$

Angle Between Two Planes

How do we find the angle between two planes?

- The angle between two planes is equal to the angle between their **normal vectors**
  - It can be found using the **scalar product** of their normal vectors
- If two planes Π_1 and Π_2 with normal vectors n_1 and n_2 meet at an angle then the two planes and the two normal vectors will form a quadrilateral
  - The angles between the planes and the normal will both be 90°
  - The angle between the two planes and the angle opposite it (between the two normal vectors) will add up to 180°



Exam Tip

- In your exam read the question carefully to see if you need to find the acute or obtuse angle
  - When revising, get into the practice of double checking at the end of a question whether your angle is acute or obtuse and whether this fits the question

Worked example

Find the acute angle between the two planes which can be defined by equations
Π_1: $2x - y + 3z = 7$ and Π_2: $x + 2y - z = 20$.

Find the normal vectors of each of the planes:

Π_1: $2x - y + 3z = 7 \Rightarrow$ normal vector, $n_1 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

Π_2: $x + 2y - z = 20 \Rightarrow$ normal vector, $n_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

Find the angle between the two normal vectors:

Angle between two vectors	$\cos\theta = \dfrac{v_1w_1 + v_2w_2 + v_3w_3}{\lvert v \rvert \lvert w \rvert}$

$$\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{|(2)(1) + (-1)(2) + (3)(-1)|}{\sqrt{2^2 + (-1)^2 + 3^2} \times \sqrt{1^2 + 2^2 + (-1)^2}} = \frac{|-3|}{\sqrt{14} \times \sqrt{6}}$$

Using the absolute value ensures we find the acute angle.

$$\theta = \cos^{-1}\left(\frac{3}{2\sqrt{21}}\right)$$

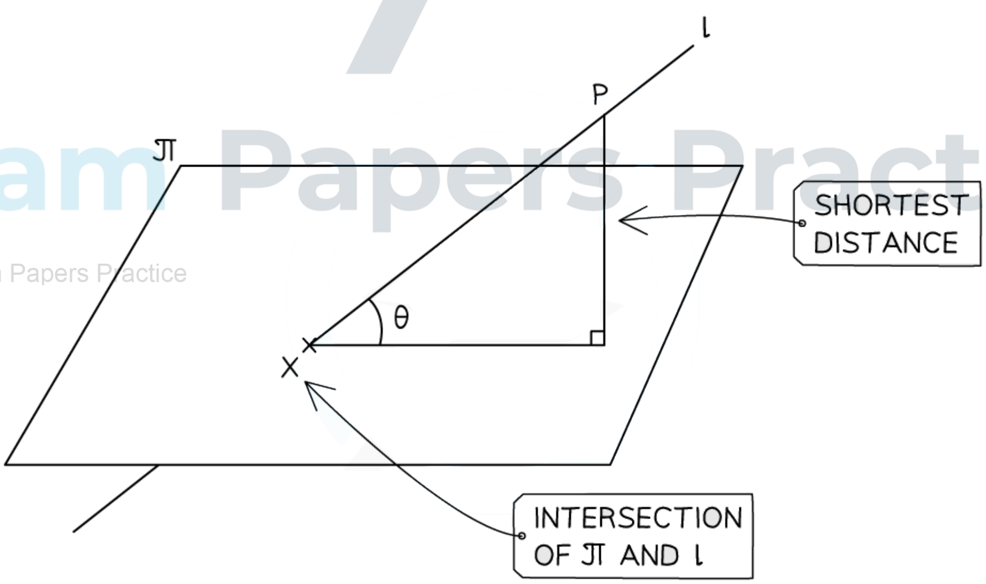
$$\boxed{\theta = 1.24 \text{ radians } (3 \text{ s.f.})}$$

3.11.4 Shortest Distances with Planes

Shortest Distance Between a Line and a Plane

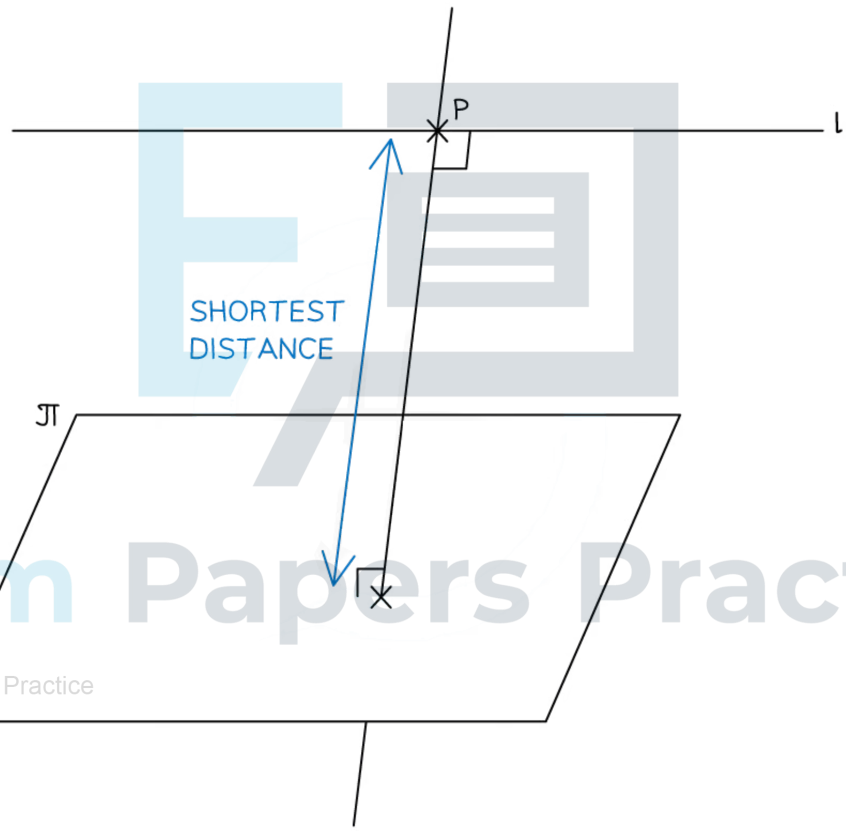
How do I find the shortest distance between a given point on a line and a plane?

- The shortest distance from any point on a line to a plane will always be the **perpendicular** distance from the point to the plane
- Given a point, *P*, on the line l with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and a plane Π with equation $\mathbf{r}\cdot\mathbf{n} = d$
  - STEP 1: Find the vector equation of the line perpendicular to the plane that goes through the point, *P*, on l
    - This will have the position vector of the point, *P*, and the direction vector **n**
  - STEP 2: Find the coordinates of the point of intersection of this new line with Π by substituting the equation of the line into the equation of the plane
  - STEP 3: Find the distance between the given point on the line and the point of intersection
    - This will be the shortest distance from the plane to the point
- A question may provide the acute angle between the line and the plane
  - Use right-angled trigonometry to find the perpendicular distance between the point on the line and the plane
    - Drawing a clear diagram will help



How do I find the shortest distance between a plane and a line parallel to the plane?

- The shortest distance between a line and a plane that are parallel to each other will be the **perpendicular** distance from the line to the plane
- Given a line l_1 with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and a plane Π parallel to l_1 with equation $\mathbf{r} \cdot \mathbf{n} = d$
  - Where **n** is the **normal vector** to the plane
  - STEP 1: Find the equation of the line l_2 perpendicular to l_1 and Π going through the point **a** in the form $\mathbf{r} = \mathbf{a} + \mu\mathbf{n}$
  - STEP 2: Find the point of intersection of the line l_2 and Π
  - STEP 3: Find the distance between the point of intersection and the point,



Exam Tip

- Vector planes questions can be tricky to visualise, read the question carefully and sketch a very simple diagram to help you get started

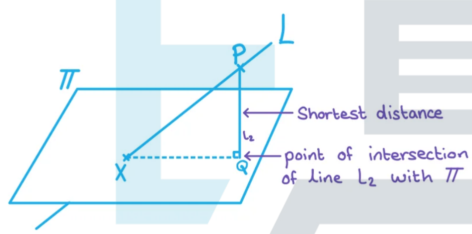
Worked example

The plane Π has equation $\mathbf{r}\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}=6$.

The line L has equation $\mathbf{r}=\begin{pmatrix}2\\3\\1\end{pmatrix}+s\begin{pmatrix}1\\-2\\4\end{pmatrix}$.

The point $P\,(-2,\ 11,\ -15)$ lies on the line L.

Find the shortest distance between the point P and the plane Π.



STEP 1: Use the given point, P and the known normal to the plane, $\underline{n}$ to write an equation for the line perpendicular to Π, L_2.

$$\underset{P}{r=\begin{pmatrix}-2\\11\\-15\end{pmatrix}}+\lambda\underset{\underline{n}}{\begin{pmatrix}2\\-1\\1\end{pmatrix}}$$

STEP 2: Find the point of intersection, Q, of the new line, L_2, with Π.

$$\left(\begin{pmatrix}-2\\11\\-15\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}\right)\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}=6$$

$$2(-2+2\lambda)-(11-\lambda)+(\lambda-15)=6$$

$$-4+4\lambda-11+\lambda+\lambda-15=6$$

$$6\lambda-30=6$$

$$\lambda=6\Rightarrow\overrightarrow{OQ}=\begin{pmatrix}-2\\11\\-15\end{pmatrix}+6\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}10\\5\\-9\end{pmatrix}$$

STEP 3: Find the distance between P and Q.

$$|\overrightarrow{PQ}|=\sqrt{(10--2)^2+(5-11)^2+(-9--15)^2}=6\sqrt{6}\text{ units}$$

$$\boxed{\text{Shortest distance}=6\sqrt{6}\text{ units}}$$

Shortest Distance Between Two Planes

How do I find the shortest distance between two parallel planes?

- Two **parallel** planes will never intersect
- The shortest distance between two **parallel planes** will be the **perpendicular distance** between them
- Given a plane Π_1 with equation $\mathbf{r} \cdot \mathbf{n} = d$ and a plane Π_2 with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ then the shortest distance between them can be found
  - STEP 1: The equation of the line perpendicular to both planes and through the point **a** can be written in the form **r** = **a** + s**n**
  - STEP 2: Substitute the equation of the line into $\mathbf{r} \cdot \mathbf{n} = d$ to find the coordinates of the point where the line meets Π_1
  - STEP 3: Find the distance between the two points of intersection of the line with the two planes

How do I find the shortest distance from a given point on a plane to another plane?

- The shortest distance from any point, P on a plane, Π_1, to another plane, Π_2 will be the **perpendicular** distance from the point to Π_2
  - STEP 1: Use the given coordinates of the point P on Π_1 and the normal to the plane Π_2 to find the vector equation of the line through P that is perpendicular to Π_1
  - STEP 2: Find the point of intersection of this line with the plane Π_2
  - STEP 3: Find the distance between the two points of intersection

Exam Tip

- There are a lot of steps when answering these questions so set your methods out clearly in the exam

Worked example

Consider the parallel planes defined by the equations:

$$\Pi_1 : \mathbf{r} \cdot \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = 44,$$

$$\Pi_2 : \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Find the shortest distance between the two planes Π_1 and Π_2.

Find the equation of the line perpendicular to the planes through the point $(0,0,3)$

position vector of Π_2

$$L: \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + s\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$$

Normal vector of Π_1

Substitute the equation of L into the equation of Π_1:

$$\begin{pmatrix} 3s \\ -5s \\ 3+2s \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = 44$$

$$3(3s) - 5(-5s) + 2(3+2s) = 44$$

$$38s + 6 = 44$$

$$s = 1$$

Substitute $s = 1$ back into the equation of L:

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \\ 5 \end{pmatrix}$$

Find the distance between $(0,0,3)$ and $(3,-5,5)$

$$d = \sqrt{3^2 + (-5)^2 + (5-3)^2}$$

$$= \sqrt{38}$$

$\boxed{\text{Shortest distance} = \sqrt{38} \text{ units}}$