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3.10 Vector Equations of Lines

IB Maths - Revision Notes

AA HL

3.10.1 Vector Equations of Lines

Equation of a Line in Vector Form

How do I find the vector equation of a line?

- The formula for finding the **vector equation** of a line is
  - $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$
    - Where $\boldsymbol{r}$ is the **position vector** of any point on the line
    - $\boldsymbol{a}$ is the **position vector** of a known point on the line
    - $\boldsymbol{b}$ is a **direction** (displacement) **vector**
    - λ is a scalar
  - This is **given in the formula booklet**
  - This equation can be used for vectors in both 2- and 3- dimensions
- This formula is similar to a regular equation of a straight line in the form $y = mx + c$ but with a vector to show both a point on the line and the direction (or gradient) of the line
  - In 2D the gradient can be found from the direction vector
  - In 3D a numerical value for the direction cannot be found, it is given as a vector
- As $\boldsymbol{a}$ could be the position vector of **any** point on the line and $\boldsymbol{b}$ could be **any scalar multiple** of the direction vector there are infinite vector equations for a single line
- Given any two points on a line with position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ the **displacement** vector can be written as $\boldsymbol{b} - \boldsymbol{a}$
  - So the formula $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$ can be used to find the vector equation of the line
  - This is **not given in the formula booklet**

How do I determine whether a point lies on a line?

- Given the equation of a line $\boldsymbol{r} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix} + \lambda \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{pmatrix}$ the point $\boldsymbol{c}$ with position vector $\begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \boldsymbol{c}_3 \end{pmatrix}$ is on the line if there exists a value of λ such that
  - $\begin{pmatrix} \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \boldsymbol{c}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{pmatrix} + \lambda \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{pmatrix}$
  - This means that there exists a single value of λ that satisfies the three equations:
    - $c_1 = a_1 + \lambda b_1$

- $c_2 = a_2 + \lambda b_2$
- $c_3 = a_3 + \lambda b_3$

- A GDC can be used to solve this system of linear equations for
  - The point only lies on the line if a single value of λ exists for all three equations
- Solve one of the equations first to find a value of λ that satisfies the first equation and then check that this value also satisfies the other two equations
- If the value of λ does not satisfy all three equations, then the point $\boldsymbol{c}$ does not lie on the line

Exam Tip

- Remember that the vector equation of a line can take many different forms
  - This means that the answer you derive might look different from the answer in a mark scheme
- You can choose whether to write your vector equations of lines using unit vectors or as column vectors
  - Use the form that you prefer, however column vectors is generally easier to work with

Worked example

a) Find a vector equation of a straight line through the points with position vectors **a** = 4**i** - 5**k** and **b** = 3**i** - 3**k**

Use the position vectors to find the displacement vector between them.

$\vec{OA} = \begin{pmatrix}4\\0\\-5\end{pmatrix}, \quad \vec{OB} = \begin{pmatrix}3\\0\\-3\end{pmatrix} \Rightarrow \vec{AB} = \begin{pmatrix}3\\0\\-3\end{pmatrix} - \begin{pmatrix}4\\0\\-5\end{pmatrix} = \begin{pmatrix}-1\\0\\2\end{pmatrix}$

Vector equation of a line	$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$

$r = \begin{pmatrix}4\\0\\-5\end{pmatrix} + \lambda\begin{pmatrix}-1\\0\\2\end{pmatrix}$ or $r = \begin{pmatrix}3\\0\\-3\end{pmatrix} + \lambda\begin{pmatrix}-1\\0\\2\end{pmatrix}$

(position vector of point a; direction vector) (position vector of point b; direction vector)

$$r = \begin{pmatrix}4\\0\\-5\end{pmatrix} + \lambda\begin{pmatrix}-1\\0\\2\end{pmatrix}$$

b) Determine whether the point C with coordinate (2, 0, -1) lies on this line.

Let $c = \begin{pmatrix}2\\0\\-1\end{pmatrix}$, then check to see if there exists a value of λ such that

$$\begin{pmatrix}2\\0\\-1\end{pmatrix} = \begin{pmatrix}4\\0\\-5\end{pmatrix} + \lambda\begin{pmatrix}-1\\0\\2\end{pmatrix}$$

From the 'i' component: $4 - \lambda = 2$ ①

From the 'j' component: $0 + 0\lambda = 0$ ② (✓) Works for all λ

From the 'k' component: $-5 + 2\lambda = -1$ ③

① $\Rightarrow \lambda = 2$ sub into ③ $\Rightarrow -5 + (2 \times 2) = -5 + 4 = -1$ ✓

Point C lies on the line

Equation of a Line in Parametric Form

How do I find the vector equation of a line in parametric form?

- By considering the three separate components of a vector in the *x*, *y* and *z* directions it is possible to write the **vector equation** of a line as **three separate equations**
  - Letting $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ then $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$ becomes
  - $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$
    - Where $\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ is a position vector and $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ is a direction vector
  - This vector equation can then be split into its three separate component forms:
    - $x = x_0 + \lambda l$
    - $y = y_0 + \lambda m$
    - $z = z_0 + \lambda n$
  - These are **given in the formula booklet**

Worked example

Write the parametric form of the equation of the line which passes through the point (-2, 1, 0) with direction vector $\begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$.

Parametric form of the equation of a line	$x = x_0 + \lambda l,\ y = y_0 + \lambda m,\ z = z_0 + \lambda n$

Use $r = a + \lambda b$ to write the equation in vector form first:

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$$

position vector of a point ↑   ↖ direction vector

Separate the components into their 3 separate equations.

$$x = -2 + 3\lambda$$
$$y = 1 + \lambda$$
$$z = -4\lambda$$

Equation of a Line in Cartesian Form

- The **Cartesian** equation of a line can be found from the **vector equation of a line** by
  - Finding the vector equation of the line in parametric form
  - Eliminating λ from the parametric equations
    - λ can be eliminated by **making it the subject** of each of the parametric equations
    - For example: $x = x_0 + \lambda l$ gives $\lambda = \frac{x - x_0}{l}$
- In **2D** the **cartesian equation of a line** is a regular equation of a straight line simply given in the form
  - $y = mx + c$
  - $ax + by + d = 0$
  - $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$ by rearranging $y - y_1 = m(x - x_1)$
- In **3D** the **cartesian equation of a line** also includes *z* and is given in the form
  - $\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$
  - where $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$
  - This is **given in the formula booklet**
- If one of your variables **does not depend on** λ then this part can be written as a separate equation
  - For example: $m = 0 \Rightarrow y = y_0$ gives $\frac{x - x_0}{l} = \frac{z - z_0}{n}$, $y = y_0$

How do I find the vector equation of a line given the Cartesian form?

- If you are given the **Cartesian** equation of a line in the form
  - $\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$
- A vector equation of the line can be found by
  - STEP 1: Set each part of the equation equal to λ individually
  - STEP 2: Rearrange each of these three equations (or two if working in 2D) to make *x*, *y*, and *z* the subjects
    - This will give you the three **parametric equations**

- $x = x_0 + \lambda l$
- $y = y_0 + \lambda m$
- $z = z_0 + \lambda n$

- STEP 3: Write this in the vector form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}$
- STEP 4: Set r to equal $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$
- If one part of the cartesian equation is given separately and is not in terms of λ then the corresponding component in the direction vector is equal to zero

Worked example

A line has the vector equation $r = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$. Find the Cartesian equation of the line.

Cartesian equations of a line	$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$

Begin by writing the equation of the line in parametric form:

$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} \Rightarrow$
$x = 1 + 4\lambda$ ①
$y = -2\lambda$ ②
$z = 2 + \lambda$ ③

Rearrange each equation to make λ the subject:

① $\lambda = \frac{x - 1}{4}$

② $\lambda = \frac{y}{-2}$

③ $\lambda = z - 2$

Set each expression for λ equal to each other:

$$\frac{x - 1}{4} = \frac{y}{-2} = z - 2$$

3.10.2 Applications to Kinematics

Kinematics using Vectors

How are vectors related to kinematics?

- Vectors are often used in questions in the context of forces, acceleration or velocity
- If an object is moving in **one dimension** then its velocity, displacement and time are related using the formula $s = vt$
  - where s is **displacement**, v is **velocity** and t is the **time taken**
- If an object is moving in **more than one dimension** then **vectors** are needed to represent its **velocity** and **displacement**
  - Whilst **time** is a **scalar quantity**, **displacement** and **velocity** are both **vector quantities**
- For an object moving at a **constant speed** in a **straight line** its velocity, displacement and time can be related using the vector equation of a line
  - $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$
  - Letting
    - $\boldsymbol{r}$ be the position of the object at the time, t
    - $\boldsymbol{a}$ be the position vector, $\boldsymbol{r}_0$ at the start ($t = 0$)
    - λ represent the time, t
    - $\boldsymbol{b}$ be the **velocity** vector, $\boldsymbol{v}$
  - Then the position of the object at the time, t can be given by
    - $\boldsymbol{r} = \boldsymbol{r}_0 + t\mathbf{v}$
  - The speed of the object will be the magnitude of the velocity $|\boldsymbol{v}|$

Exam Tip

- Kinematics questions can have a lot of information in, read them carefully and pick out the parts that are essential to the question
- Look out for where variables used are the same and/or different within vector equations, you will need to use different techniques to find these

Worked example

A car, moving at constant speed, takes 2 minutes to drive in a straight line from point A (-4, 3) to point B (6, -5).

At time t, in minutes, the position vector ($\boldsymbol{p}$) of the car relative to the origin can be given in the form $\boldsymbol{p} = \boldsymbol{a} + t\boldsymbol{b}$.

Find the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Vector $\underline{a}$ represents the initial position and vector $\underline{b}$ represents the direction vector per minute.

Position vector $\overrightarrow{OA} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

At $t = 0$ minutes, $\underline{p} = \underline{a}$ so $\underline{a} = \overrightarrow{OA} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

Position vector $\overrightarrow{OB} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$

At $t = 2$ minutes, the car is at the point B and so $\overrightarrow{OB} = \underline{a} + 2\underline{b}$

$\begin{pmatrix} 6 \\ -5 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} + 2\underline{b}$

Direction vector $2\underline{b} = \begin{pmatrix} 6 \\ -5 \end{pmatrix} - \begin{pmatrix} -4 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ -8 \end{pmatrix}$

$\underline{a} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$

3.10.3 Pairs of Lines in 3D

Coincident, Parallel, Intersecting & Skew Lines

How do I tell if two lines are parallel?

- Two lines are parallel if, and only if, their **direction vectors** are **parallel**
  - This means the direction vectors will be **scalar multiples** of each other
  - For example, the lines whose equations are $\mathbf{r}=\begin{pmatrix}2\\1\\-7\end{pmatrix}+\lambda_1\begin{pmatrix}2\\0\\-8\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}1\\-1\\5\end{pmatrix}+\lambda_2\begin{pmatrix}-1\\0\\4\end{pmatrix}$ are parallel
    - This is because $\begin{pmatrix}2\\0\\-8\end{pmatrix}=-2\begin{pmatrix}-1\\0\\4\end{pmatrix}$

How do I tell if two lines are coincident?

- **Coincident lines** are two lines that lie directly on top of each other
  - They are indistinguishable from each other
- Two parallel lines will either **never intersect** or they are **coincident (identical)**
  - Sometimes the vector equations of the lines may look different
    - for example, the lines represented by the equations $\mathbf{r}=\begin{pmatrix}1\\-8\end{pmatrix}+s\begin{pmatrix}-4\\8\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}-3\\0\end{pmatrix}+t\begin{pmatrix}1\\-2\end{pmatrix}$ are **coincident**,
  - To check whether two lines are **coincident**:
    - First check that they are **parallel**
      - They are because $\begin{pmatrix}-4\\8\end{pmatrix}=-4\begin{pmatrix}1\\-2\end{pmatrix}$ and so their direction vectors are parallel
    - Next, determine whether **any point** on one of the lines also lies on the other

- $\begin{pmatrix} 1 \\ -8 \end{pmatrix}$ is the position vector of a point on the first line and
  $\begin{pmatrix} 1 \\ -8 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ so it also lies on the second line
- If two parallel lines share **any point**, then they share **all points** and are **coincident**

What are skew lines?

- Lines that are **not parallel** and which **do not intersect** are called **skew lines**
  - This is only possible in **3-dimensions**

How do I determine whether lines in 3 dimensions are parallel, skew, or intersecting?

- First, look to see if the direction vectors are parallel:
  - if the **direction vectors are parallel**, then the **lines are parallel**
  - if the **direction vectors are not parallel**, the **lines are not parallel**
- If the lines are **parallel**, check to see if the lines are **coincident**:
  - If they **share any point**, then they are **coincident**
  - If **any point** on one line is **not on the other line**, then the lines are **not coincident**
- If the lines are **not parallel**, check whether they **intersect**:
  - STEP 1: Set the vector equations of the two lines equal to each other with **different variables**
    - e.g. λ and μ, for the parameters
  - STEP 2: Write the three separate equations for the **i**, **j**, and **k** components in terms of λ and μ
  - STEP 3: **Solve** two of the equations to find a value for λ and μ
  - STEP 4: **Check** whether the values of λ and μ you have found satisfy the third equation
    - If **all three** equations are satisfied, then the lines **intersect**
    - If **not all three** equations are satisfied, then the lines are **skew**

How do I find the point of intersection of two lines?

- If a pair of lines are **not parallel** and **do intersect**, a unique point of intersection can be found
  - If the two lines intersect, there will be a single point that will lie on both lines
- Follow the steps above to find the values of λ and μ that satisfy **all three equations**
  - STEP 5: Substitute either the value of λ or the value of μ into one of the vector equations to find the position vector of the point where the lines intersect
  - It is always a good idea to **check** in the other equations as well, you should get the same point for each line

Exam Tip

- Make sure that you use different letters, e.g. λ and μ, to represent the parameters in vector equations of different lines
  - Check that the variable you are using has not already been used in the question

Worked example

Determine whether the following pair of lines are parallel, intersect, or are skew.

$\mathbf{r} = 4\mathbf{i} + 3\mathbf{j} + s(5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = -5\mathbf{i} + 4\mathbf{j} + \mathbf{k} + t(2\mathbf{i} - \mathbf{j})$.

STEP 1: Check to see if the lines are parallel:

$$r_1 = \begin{pmatrix}4\\3\\0\end{pmatrix} + \lambda\begin{pmatrix}5\\2\\3\end{pmatrix} \qquad r = \begin{pmatrix}-5\\4\\1\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\0\end{pmatrix}$$

direction vectors

The lines are not parallel because there is no value of k such that $\begin{pmatrix}5\\2\\3\end{pmatrix} = k\begin{pmatrix}2\\-1\\0\end{pmatrix}$

STEP 2: Check to see if the lines intersect:

$4 + 5\lambda = -5 + 2\mu$ ①

$3 + 2\lambda = 4 - \mu$ ②

$3\lambda = 1$ ③

Set up three equations for each of the i, j and k components.

Equation ③: $\lambda = \frac{1}{3}$ Sub into ②: $3 + 2\left(\frac{1}{3}\right) = 4 - \mu$

$\frac{11}{3} = 4 - \mu$

$\mu = \frac{1}{3}$

Sub into ①: $4 + 5\left(\frac{1}{3}\right) = -5 + 2\left(\frac{1}{3}\right)$

$\frac{17}{3} \neq -\frac{13}{3}$ contradiction

There is no point of intersection.

The lines are skew

Angle Between Two Lines

How do we find the angle between two lines?

- The angle between two lines is equal to the angle between their **direction vectors**
  - It can be found using the **scalar product** of their direction vectors
- Given two lines in the form $\boldsymbol{r} = \boldsymbol{a}_1 + \lambda \boldsymbol{b}_1$ and $\boldsymbol{r} = \boldsymbol{a}_2 + \lambda \boldsymbol{b}_2$ use the formula
  - $\theta = \cos^{-1}\left(\dfrac{\boldsymbol{b}_1 \bullet \boldsymbol{b}_2}{|\boldsymbol{b}_1||\boldsymbol{b}_2|}\right)$
- If you are given the equations of the lines in a different form or two points on a line you will need to find their direction vectors first
- To find the angle ABC the vectors BA and BC would be used, both starting from the point B
- The intersection of two lines will always create **two angles**, an acute one and an obtuse one
  - These two angles will add to 180°
  - You may need to subtract your answer from 180° to find the angle you are looking for
  - A **positive scalar product** will result in the **acute angle** and a **negative scalar product** will result in the **obtuse angle**
    - Using the absolute value of the scalar product will always result in the acute angle

Exam Tip

- In your exam read the question carefully to see if you need to find the acute or obtuse angle
  - When revising, get into the practice of double checking at the end of a question whether your angle is acute or obtuse and whether this fits the question

Worked example

Find the acute angle, in radians between the two lines defined by the equations:

$$l_1:\ \boldsymbol{a} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} \text{ and } l_2:\ \boldsymbol{b} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix}$$

STEP 1: Find the scalar product of the direction vectors:

$$\begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ 5 \end{pmatrix} = (1 \times -3) + (-4 \times 2) + (-3 \times 5) = -3 + (-8) + (-15) = -26$$

negative, so the angle will be the obtuse angle.

STEP 2: Find the magnitudes of the direction vectors:

$$\sqrt{(1)^2 + (-4)^2 + (-3)^2} = \sqrt{26} \qquad \sqrt{(-3)^2 + (2)^2 + (5)^2} = \sqrt{38}$$

STEP 3: Find the angle: $\cos\theta = \dfrac{|-26|}{\sqrt{26}\sqrt{38}}$

Using the absolute value will result in the acute angle

$$\theta = \cos^{-1}\left(\frac{26}{\sqrt{26}\sqrt{38}}\right)$$

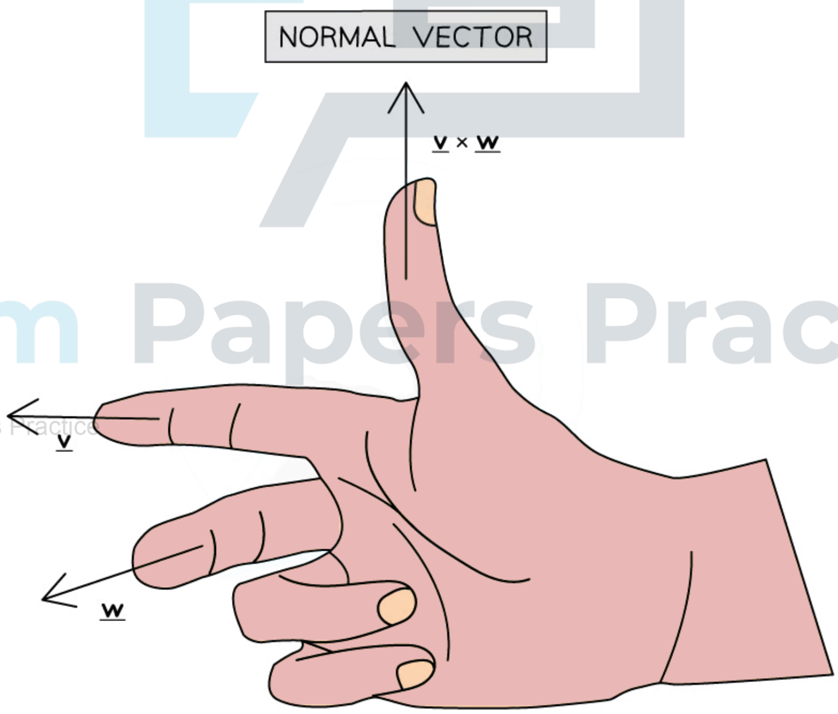
$\theta = 0.597$ radians (3sf)

3.10.4 The Vector Product

The Vector ('Cross') Product

What is the vector (cross) product?

- The **vector product** (also known as the **cross** product) is a form in which two vectors can be combined together
- The vector product between two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is denoted $\boldsymbol{v} \times \boldsymbol{w}$
- The result of taking the vector product of two vectors is a **vector**
- The **vector product** is a vector **in a plane** that is **perpendicular** to the two vectors from which it was calculated
    - This could be in either direction, depending on the angle between the two vectors
    - The **right-hand** rule helps you see which direction the vector product goes in
        - By pointing your index finger and your middle finger in the direction of the two vectors your thumb will automatically go in the direction of the vector product



How do I find the vector (cross) product?

- There are **two methods** for calculating the vector product

- The **vector product** of the two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ can be written in **component form** as follows:
  - $\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$
  - Where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
  - This is **given in the formula booklet**
- The vector product can also be found in terms of its **magnitude** and **direction**
- The **magnitude of the vector product** is equal to the **product of the magnitudes** of the two vectors and the **sine of the angle between them**
  - $|\boldsymbol{v} \times \boldsymbol{w}| = |v||w|\sin\theta$
  - Where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$
    - The two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are joined at the start and pointing away from each other
  - This is **given in the formula booklet**
- The **direction of the vector product** is **perpendicular** to both $\boldsymbol{v}$ and $\boldsymbol{w}$

What properties of the vector product do I need to know?

- The order of the vectors **is important** and **changes the result** of the vector product
  - $\boldsymbol{v} \times \boldsymbol{w} \neq \boldsymbol{w} \times \boldsymbol{v}$
  - However
  - $\boldsymbol{v} \times \boldsymbol{w} = -\boldsymbol{w} \times \boldsymbol{v}$
- The **distributive law** can be used to 'expand brackets'
  - $\boldsymbol{u} \times (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{w}$
  - Where $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are all vectors
- Multiplying a **scalar** by a vector gives the result:
  - $(k\boldsymbol{v}) \times \boldsymbol{w} = \boldsymbol{v} \times (k\boldsymbol{w}) = k(\boldsymbol{v} \times \boldsymbol{w})$
- The vector product between a vector and itself is equal to **zero**
  - $\boldsymbol{v} \times \boldsymbol{v} = 0$
- If two vectors are **parallel** then the vector product is **zero**
  - This is because sin 0° = sin 180° = 0
- If $\boldsymbol{v} \times \boldsymbol{w} = 0$ then $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel if they are non-zero
- If two vectors, $\boldsymbol{v}$ and $\boldsymbol{w}$, are **perpendicular** then the magnitude of the vector product is equal to the **product** of the magnitudes of the vectors
  - $|\boldsymbol{v} \times \boldsymbol{w}| = |\boldsymbol{w}||\boldsymbol{v}|$
  - This is because sin 90° = 1

Exam Tip

- The formulae for the vector product are given in the formula booklet, make sure you use them as this is an easy formula to get wrong
- The properties of the vector product are not given in the formula booklet, however they are important and it is likely that you will need to recall them in your exam so be sure to commit them to memory

Worked example

Calculate the magnitude of the vector product between the two vectors $\boldsymbol{v} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix}$ and $\boldsymbol{w} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ using

i) the formula $\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$,

$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix} \quad \underline{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$

Use the formula to find the cross-product:

$\underline{v} \times \underline{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix} = \begin{pmatrix} (0)(-1) - (-5)(-2) \\ (-5)(3) - (2)(-1) \\ (2)(-2) - (0)(3) \end{pmatrix} = \begin{pmatrix} -10 \\ -13 \\ -4 \end{pmatrix}$

Find the magnitude of $\underline{v} \times \underline{w}$:

$|\underline{v} \times \underline{w}| = \sqrt{(-10)^2 + (-13)^2 + (-4)^2} = \sqrt{285}$

$|\underline{v} \times \underline{w}| = 16.9 \text{ (3sf)}$

ii) the formula, given that the angle between them is 1 radian.

Find the magnitude of $\underline{v}$ and $\underline{w}$:

$|\underline{v}| = \sqrt{2^2 + 0^2 + (-5)^2} = \sqrt{29}$

$|\underline{w}| = \sqrt{3^2 + (-2)^2 + (-1)^2} = \sqrt{14}$

$|\underline{v} \times \underline{w}| = |\underline{v}||\underline{w}| \sin\theta$

$= \sqrt{29} \times \sqrt{14} \sin(1^c)$

$|\underline{v} \times \underline{w}| = 17.0 \text{ (3sf)}$

Areas using Vector Product

How do I use the vector product to find the area of a parallelogram?

- The **area of the parallelogram** with two adjacent sides formed by the vectors ***v*** and ***w*** is equal to **the magnitude of the vector product** of two vectors ***v*** and ***w***
  - $A = |\boldsymbol{v} \times \boldsymbol{w}|$ where ***v*** and ***w*** form two **adjacent sides** of the parallelogram
    - This is **given in the formula booklet**

How do I use the vector product to find the area of a triangle?

- The **area of the triangle** with two sides formed by the vectors ***v*** and ***w*** is equal to **half of the magnitude of the vector product** of two vectors ***v*** and ***w***
  - $A = \frac{1}{2}|\boldsymbol{v} \times \boldsymbol{w}|$ where ***v*** and ***w*** form two **sides** of the triangle
    - This is **not** given in the formula booklet

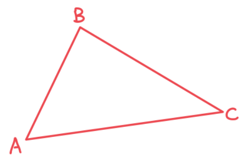
Exam Tip

- The formula for the area of the parallelogram is given in the formula booklet but the formula for the area of a triangle is not
  - Remember that the area of a triangle is half the area of a parallelogram

Worked example

Find the area of the triangle enclosed by the coordinates (1, 0, 5), (3, -1, 2) and (2, 0, -1).

Let A be (1, 0, 5), B be (3, -1, 2) and C be (2, 0, -1)



You can use any two direction vectors moving away from any vertex.

Find the two direction vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \qquad \overrightarrow{AC} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -6 \end{pmatrix}$$

Find the cross product of the two direction vectors:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -6 \end{pmatrix} = \begin{pmatrix} (-1)(-6) - (-3)(0) \\ (-3)(1) - (2)(-6) \\ (2)(0) - (-1)(1) \end{pmatrix} = \begin{pmatrix} 6 \\ 9 \\ 1 \end{pmatrix}$$

Find the magnitude of the cross product

$$\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{6^2 + 9^2 + 1^2} = \sqrt{118}$$

Area of the triangle is half the magnitude

$$\text{Area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{118}$$

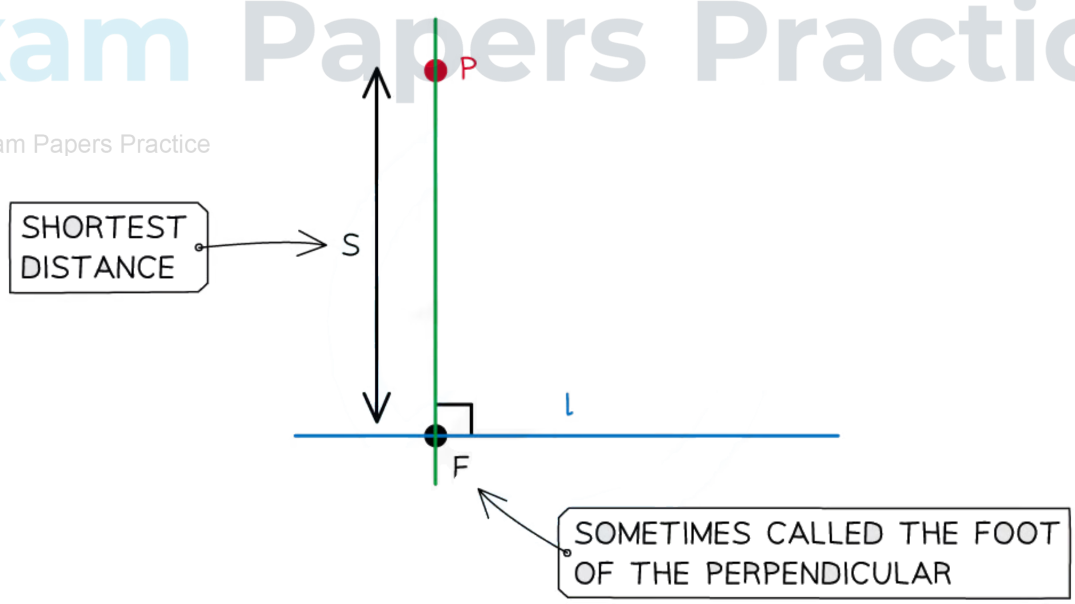
Area = 5.43 u^2 (3sf)

3.10.5 Shortest Distances with Lines

Shortest Distance Between a Point and a Line

How do I find the shortest distance from a point to a line?

- The shortest distance from any point to a line will always be the **perpendicular** distance
  - Given a line *l* with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and a point *P* not on *l*
  - The **scalar product** of the direction vector, **b**, and the vector in the direction of the **shortest distance** will be zero
- The shortest distance can be found using the following steps:
  - STEP 1: Let the vector equation of the line be ***r*** and the point not on the line be *P*, then the point on the line closest to *P* will be the point *F*
    - The point *F* is sometimes called the foot of the perpendicular
  - STEP 2: Sketch a diagram showing the line *l* and the points *P* and *F*
    - The vector $\overrightarrow{FP}$ will be **perpendicular** to the line *l*
  - STEP 3: Use the equation of the line to find the position vector of the point *F* in terms of λ
  - STEP 4: Use this to find the displacement vector $\overrightarrow{FP}$ in terms of λ
  - STEP 5: The scalar product of the direction vector of the line *l* and the displacement vector $\overrightarrow{FP}$ will be zero
    - Form an equation $\overrightarrow{FP} \cdot \mathbf{b} = 0$ and solve to find λ
  - STEP 6: Substitute λ into $\overrightarrow{FP}$ and find the magnitude $\left|\overrightarrow{FP}\right|$
    - The shortest distance from the point to the line will be the magnitude of $\overrightarrow{FP}$
- Note that the shortest distance between the point and the line is sometimes referred to as the **length of the perpendicular**



How do we use the vector product to find the shortest distance from a point to a line?

- The vector product can be used to find the shortest distance from any point to a line on a 2–dimensional plane
- Given a point, P, and a line $\boldsymbol{r} = a + \lambda b$
  - The shortest distance from P to the line will be $\dfrac{\left|\overrightarrow{AP} \times b\right|}{|b|}$
  - Where A is a point on the line
  - This is **not** given in the formula booklet

Exam Tip

- Column vectors can be easier and clearer to work with when dealing with scalar products.

Worked example

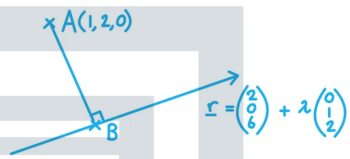
Point A has coordinates (1, 2, 0) and the line l has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

Point B lies on the l such that $[AB]$ is perpendicular to l.

Find the shortest distance from A to the line l.

B is on l so can be written in terms of λ:

$$\overrightarrow{OB} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ \lambda \\ 6+2\lambda \end{pmatrix}$$



Find $\overrightarrow{AB}$ using $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ \lambda \\ 6+2\lambda \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda - 2 \\ 6+2\lambda \end{pmatrix}$$

$\overrightarrow{AB}$ is perpendicular to l: $\overrightarrow{AB} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0$

$$\begin{pmatrix} 1 \\ \lambda - 2 \\ 6+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0$$

$$\lambda - 2 + 2(6+2\lambda) = 0$$

$$5\lambda + 10 = 0$$

$$\lambda = -2$$

Substitute back into $\overrightarrow{AB}$ and find the magnitude:

$$\overrightarrow{AB} = \begin{pmatrix} 1 \\ -2-2 \\ 6+2(-2) \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$$

$$|\overrightarrow{AB}| = \sqrt{1^2 + (-4)^2 + 2^2} = \sqrt{21}$$

Shortest distance $= \sqrt{21}$ units

Shortest Distance Between Two Lines

How do we find the shortest distance between two parallel lines?

- Two **parallel** lines will never intersect
- The shortest distance between two **parallel lines** will be the **perpendicular distance** between them
- Given a line l_1 with equation $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and a line l_2 with equation $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{d}_2$ then the shortest distance between them can be found using the following steps:
  - STEP 1: Find the vector between $\mathbf{a}_1$ and a general coordinate from l_2 in terms of μ
  - STEP 2: Set the scalar product of the vector found in STEP 1 and the direction vector $\mathbf{d}_1$ equal to zero
    - Remember the direction vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are scalar multiples of each other and so either can be used here
  - STEP 3: Form and solve an equation to find the value of μ
  - STEP 4: Substitute the value of μ back into the equation for l_2 to find the coordinate on l_2 closest to l_1
  - STEP 5: Find the distance between $\mathbf{a}_1$ and the coordinate found in STEP 4
- Alternatively, the formula $\frac{|\overrightarrow{AB} \times \mathbf{d}|}{|\mathbf{d}|}$ can be used
  - Where $\overrightarrow{AB}$ is the vector connecting the two given coordinates $\mathbf{a}_1$ and $\mathbf{a}_2$
  - **d** is the simplified vector in the direction of $\mathbf{d}_1$ and $\mathbf{d}_2$
  - This is **not** given in the formula booklet

How do we find the shortest distance from a given point on a line to another line?

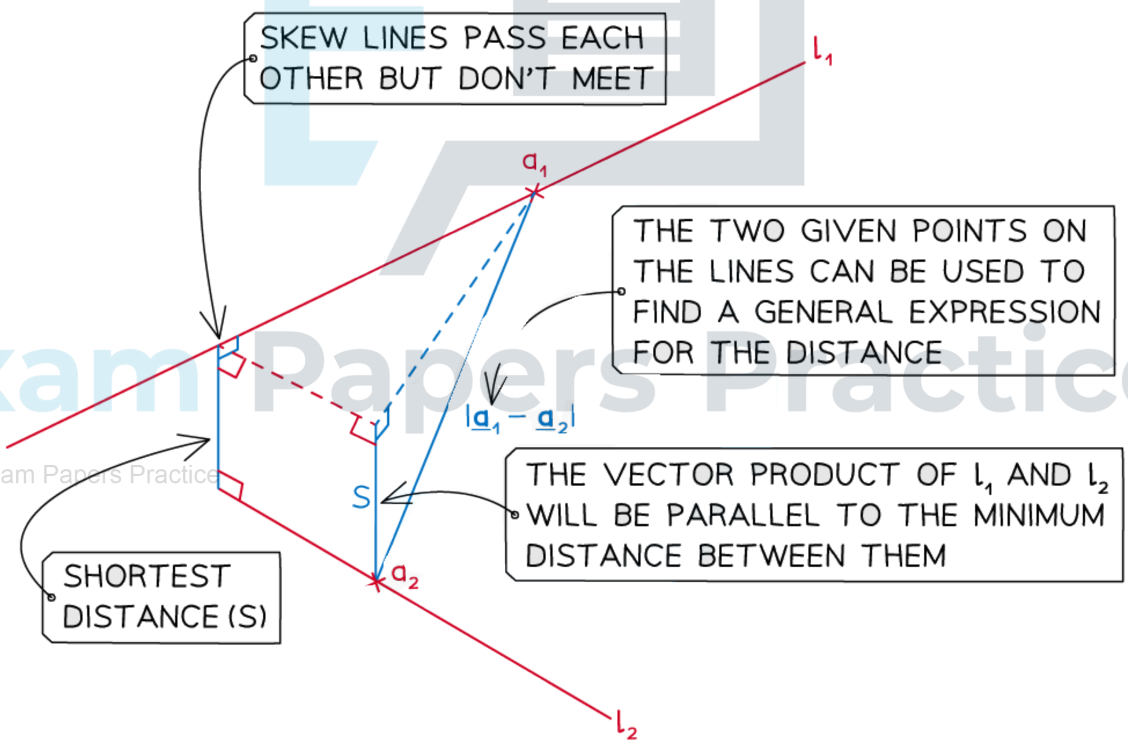
- The shortest distance from any point on a line to another line will be the **perpendicular** distance from the point to the line
- If the angle between the two lines is known or can be found then right-angled trigonometry can be used to find the perpendicular distance
  - The formula $\frac{|\overrightarrow{AB} \times \mathbf{d}|}{|\mathbf{d}|}$ given above is derived using this method and can be used
- Alternatively, the equation of the line can be used to find a general coordinate and the steps above can be followed to find the shortest distance

How do we find the shortest distance between two skew lines?

- Two **skew** lines are not parallel but will never intersect
- The shortest distance between two **skew lines** will be perpendicular to **both** of the lines

- This will be at the point where the two lines pass each other with the perpendicular distance where the point of intersection would be
- The **vector product** of the two direction vectors can be used to find a vector in the direction of the shortest distance
- The shortest distance will be a vector **parallel** to the vector product

- To find the shortest distance between two skew lines with equations $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{d}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{d}_2$,
  - STEP 1: Find the vector product of the direction vectors $\mathbf{d}_1$ and $\mathbf{d}_2$
    - $\mathbf{d} = \mathbf{d}_1 \times \mathbf{d}_2$
  - STEP 2: Find the vector in the direction of the line between the two general points on l_1 and l_2 in terms of λ and μ
    - $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
  - STEP 3: Set the two vectors parallel to each other
    - $\mathbf{d} = k\overrightarrow{AB}$
  - STEP 4: Set up and solve a system of linear equations in the three unknowns, k, λ and μ



Exam Tip

- Exam questions will often ask for the shortest, or minimum, distance within vector questions
- If you're unsure start by sketching a quick diagram
- Sometimes calculus can be used, however usually vector methods are required

Exam Papers Practice

Worked example

Consider the skew lines l_1 and l_2 as defined by:

$$l_1 : \boldsymbol{r} = \begin{pmatrix} 6 \\ -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

$$l_2 : \boldsymbol{r} = \begin{pmatrix} -5 \\ 4 \\ -8 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Find the minimum distance between the two lines.

Find the vector product of the direction vectors.

$$\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} (-3)(1)-(4)(2) \\ (4)(-1)-(2)(1) \\ (2)(2)-(-3)(-1) \end{pmatrix} = \begin{pmatrix} -11 \\ -6 \\ 1 \end{pmatrix}$$

Find the vector in the direction of the line between the general coordinates.

$$\overrightarrow{AB} = \begin{pmatrix} -5-\mu \\ 4+2\mu \\ -8+\mu \end{pmatrix} - \begin{pmatrix} 6+2\lambda \\ -4-3\lambda \\ 3+4\lambda \end{pmatrix} = \begin{pmatrix} -11-\mu-2\lambda \\ 8+2\mu+3\lambda \\ -11+\mu-4\lambda \end{pmatrix}$$

A point on l_2 A point on l_1

$$\begin{pmatrix} -11-\mu-2\lambda \\ 8+2\mu+3\lambda \\ -11+\mu-4\lambda \end{pmatrix} = k\begin{pmatrix} -11 \\ -6 \\ 1 \end{pmatrix}$$

$\overrightarrow{AB}$ is parallel to $\begin{pmatrix} -11 \\ -6 \\ 1 \end{pmatrix}$

so $\overrightarrow{AB} = k\begin{pmatrix} -11 \\ -6 \\ 1 \end{pmatrix}$

Set up and solve a system of equations.

$$11k - 2\lambda - \mu = 11$$
$$6k + 3\lambda + 2\mu = -8$$
$$\mu - 4\lambda - k = 11$$

Solve using GDC:

$k = \frac{31}{79}$ $\lambda = -\frac{238}{79}$ $\mu = -\frac{52}{79}$

Substitute back into the expression for $\overrightarrow{AB}$ and find the magnitude:

$$|\overrightarrow{AB}| = \left| \begin{pmatrix} -11-\left(-\frac{52}{79}\right)-2\left(-\frac{238}{79}\right) \\ 8+2\left(-\frac{52}{79}\right)+3\left(-\frac{238}{79}\right) \\ -11+\left(-\frac{52}{79}\right)-4\left(-\frac{238}{79}\right) \end{pmatrix} \right| = \left| \begin{pmatrix} -\frac{341}{79} \\ -\frac{186}{79} \\ \frac{31}{79} \end{pmatrix} \right| = \sqrt{\left(-\frac{341}{79}\right)^2 + \left(-\frac{186}{79}\right)^2 + \left(\frac{31}{79}\right)^2}$$

Shortest distance = 4.93 units (3 s.f.)