DP IB Maths: AI SL

3.1 Geometry Toolkit

Contents

3.1.1 Coordinate Geometry

Basic Coordinate Geometry

What are cartesian coordinates?

- **Cartesian** coordinates are basically the *x*-*y* coordinate system
  - They allow us to label where things are in a two-dimensional plane
- In the 2D cartesian system, the horizontal axis is labelled *x* and the vertical axis is labelled *y*

What can we do with coordinates?

- If we have two points with coordinates (x_1, y_1) and (x_2, y_2) then we should be able to find
  - The **midpoint** of the two points
  - The **distance** between the two points
  - The **gradient** of the line between them

How do I find the midpoint of two points?

- The midpoint is the **average (middle) point**
  - It can be found by finding the middle of the *x*-coordinates and the middle of the *y*-coordinates
- The coordinates of the midpoint will be

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

  - This is given in the formula booklet under the prior learning section at the beginning

How do I find the distance between two points?

- The distance between two points with coordinates (x_1, y_1) and (x_2, y_2) can be found using the formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

  - This is given in the formula booklet in the *prior learning* section at the beginning
- Pythagoras' Theorem $a^2 = b^2 + c^2$ is used to find the length of a line between two coordinates
- If the coordinates are labelled A and B then the line segment between them is written with the notation [AB]

How do I find the gradient of the line between two points?

- The gradient of a line between two points with coordinates (x_1, y_1) and (x_2, y_2) can be found using the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

  - This is given in the formula booklet under section 2.1 Gradient formula
  - This is usually known as $m = \frac{\text{rise}}{\text{run}}$

Worked example

Point *A* has coordinates (3, -4) and point *B* has coordinates (-5, 2).

i) Calculate the distance of the line segment *AB*.

$A: (3, -4) \quad B: (-5, 2)$

$x_1 \quad y_1 \quad x_2 \quad y_2$

Formula for distance between two points:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Sub coordinates for A and B into the formula:

$$d = \sqrt{(3-(-5))^2 + (-4-2)^2}$$

$$= \sqrt{8^2 + (-6)^2} = \sqrt{100}$$

$$\boxed{d = 10 \text{ units}}$$

ii) Find the gradient of the line connecting points *A* and *B*.

$A:(3, -4)$ $B:(-5, 2)$

x_1 y_1 x_2 y_2

Formula for gradient of a line segment:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Sub coordinates for A and B into the formula:

$$m = \frac{2 - -4}{-5 - 3} = \frac{6}{-8} = -\frac{3}{4}$$

$$\boxed{m = -\frac{3}{4}}$$

iii) Find the midpoint of [AB].

$A:(3, -4)$ $B:(-5, 2)$

x_1 y_1 x_2 y_2

Formula for the midpoint of two coordinates:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Sub values in:

$$\text{Midpoint} = \left(\frac{3 + (-5)}{2}, \frac{-4 + 2}{2}\right) = (-1, -1)$$

$$\boxed{\text{Midpoint} = (-1, -1)}$$

Perpendicular Bisectors

What is a perpendicular bisector?

- A perpendicular bisector of a line segment cuts the line segment in half at a right angle
  - Perpendicular lines meet at right angles
  - Bisect means to cut in half
- Two lines are perpendicular if the **product of their gradients is -1**

How do I find the equation of the perpendicular bisector of a line segment?

- To find the equation of a straight line you need to find
  - The gradient of the line
  - A coordinate of a point on the line
- To find the equation of the **perpendicular bisector** of a line segment follow these steps:
  - STEP 1: Find the coordinates of the midpoint of the line segment
    - We know that the perpendicular bisector will cut the line segment in half so we can use the midpoint of the line segment as the known coordinate on the bisector
  - STEP 2: Find the gradient of the line segment
  - STEP 3: Find the gradient of the perpendicular bisector
    - This will be -1 divided by the gradient of the line segment
  - STEP 4: Substitute the gradient of the perpendicular bisector and the coordinates of the midpoint into an equation for a straight line
    - The **point-gradient** form $y - y_1 = m(x - x_1)$ is the easiest
  - STEP 5: Rearrange into the required form
    - Either $y = mx + c$ or $ax + by + d = 0$
    - These equations for a straight line are given in the formula booklet

Worked example

Point A has coordinates (4, -6) and point B has coordinates (8, 6). Find the equation of the perpendicular bisector to [AB]. Give your answer in the form $ax + by + d = 0$.

Step 1: find the coordinates of the midpoint:

$A: (4, -6)$ $B: (8, 6)$

x_1, y_1, x_2, y_2

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Sub values in:

$$\text{Midpoint} = \left(\frac{4+8}{2}, \frac{-6+6}{2}\right) = (6, 0)$$

Step 2: Find the gradient of [AB]:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - -6}{8 - 4} = \frac{12}{4} = 3$$

Step 3: Find the gradient of the perpendicular bisector:

$$m_{\perp r} = -\frac{1}{m} = -\frac{1}{3}$$

Step 4: Substitute gradient and coordinate into an equation for a straight line.

insert coordinates of the midpoint.

$$(y - y_1) = m(x - x_1)$$

$$(y - 0) = -\frac{1}{3}(x - 6)$$

Step 5: Rearrange into the form $ax + by + d = 0$

$$(y - 0) = -\frac{1}{3}(x - 6) \quad (\times -3)$$

$$-3y = x - 6 \quad (+3y)$$

$$\boxed{x + 3y - 6 = 0}$$

3.1.2 Arcs & Sectors

Length of an Arc

What is an arc?

- An arc is a part of the **circumference** of a circle
  - It is easiest to think of it as the crust of a single slice of pizza
- The length of an arc depends of the size of the angle at the centre of the circle
- If the angle at the centre is **less than 180°** then the arc is known as a **minor arc**
  - This could be considered as the crust of a single slice of pizza
- If the angle at the centre is **more than 180°** then the arc is known as a **major arc**
  - This could be considered as the crust of the remaining pizza after a slice has been taken away

How do I find the length of an arc?

- The length of an arc is simply a fraction of the circumference of a circle
  - The fraction can be found by dividing the angle at the centre by 360°
- The formula for the length, l, of an arc is

$$l = \frac{\theta}{360} \times 2\pi r$$

  - Where θ is the angle measured in degrees
  - r is the radius
  - This is **in the formula booklet**, you do not need to remember it

Worked example

A circular pizza has had a slice cut from it, the angle of the slice that was cut was 38 °. The radius of the pizza is 12 cm. Find

i) the length of the outside crust of the slice of pizza (the minor arc),

A diagram will help:

Formula for the Length of an arc:

$$l = \frac{\theta}{360} \times 2\pi r$$

Substitute:

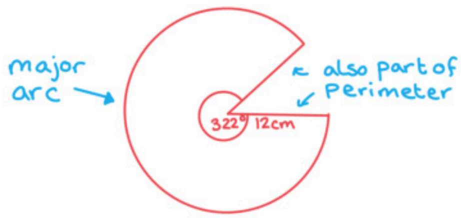
$$l = \frac{38}{360} \times 2\pi(12)$$

$$= \frac{38\pi}{15} = 7.9587\ldots \text{ cm}$$

$$\boxed{\text{Length of crust} = 7.96 \text{ cm (3s.f)}}$$

ii) the perimeter of the remaining pizza.

A diagram will help:



Formula for the Length of an arc:

$$l = \frac{\theta}{360} \times 2\pi r$$

Substitute:

$$l = \frac{322}{360} \times 2\pi(12)$$

$$= \frac{322}{15}\pi$$ ← Length of major arc

Find perimeter:

P = major arc + radius + radius

$$= \frac{322\pi}{15} + 12 + 12 = 91.4395\ldots \text{ cm}$$

$$\boxed{\text{Perimeter} = 91.4 \text{ cm (3s.f)}}$$

Area of a Sector

What is a sector?

- A sector is a part of a circle enclosed by two radii (radiuses) and an arc
  - It is easier to think of this as the shape of a single slice of pizza
- The area of a sector depends of the size of the angle at the centre of the sector
- If the angle at the centre is **less than 180°** then the sector is known as a **minor sector**
  - This could be considered as the shape of a single slice of pizza
- If the angle at the centre is **more than 180°** then the sector is known as a **major sector**
  - This could be considered as the shape of the remaining pizza after a slice has been taken away

How do I find the area of a sector?

- The area of a sector is simply a fraction of the area of the whole circle
  - The fraction can be found by dividing the angle at the centre by 360°
- The formula for the area, A, of a sector is

$$A = \frac{\theta}{360} \times \pi r^2$$

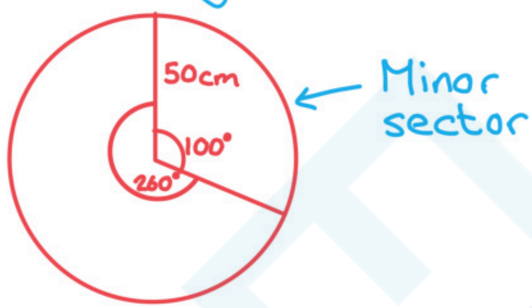
  - Where θ is the angle measured in degrees
  - r is the radius
  - This is **in the formula booklet**, you do not need to remember it

Worked example

Jamie has divided a circle of radius 50 cm into two sectors; a minor sector of angle 100° and a major sector of angle 260°. He is going to paint the minor sector blue and the major sector yellow. Find

i) the area Jamie will paint blue,

Start with a diagram:



Formula for the area of a sector:

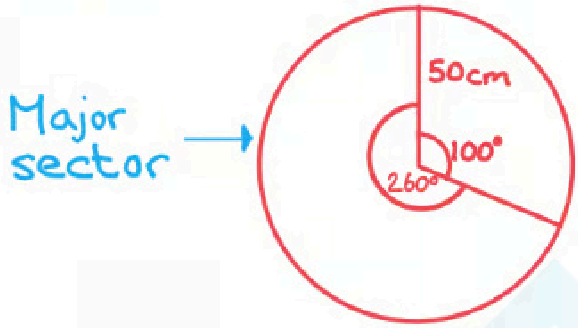
$$A = \frac{\theta}{360°} \times \pi r^2$$

Substitute:
$$A = \frac{100}{360} \times \pi \times 50^2$$
$$= \frac{6250}{9}\pi$$
$$= 2181.66\ldots \text{ cm}^2$$

$$\boxed{\text{Blue area} = 2180 \text{ cm}^2 \text{ (3sf)}}$$

ii) the area Jamie will paint yellow.

Start with a diagram:



Formula for the area of a sector:

$$A = \frac{\theta}{360°} \times \pi r^2$$

Substitute: $A = \frac{260}{360} \times \pi \times 50^2$

$$= \frac{16250}{9}\pi$$

$$= 5672.32\ldots \text{cm}^2$$

$\boxed{\text{Yellow area} = 5670 \text{ cm}^2 \text{ (3sf)}}$