# DP IB Maths: AI HL

## 3.1 Geometry Toolkit

### Contents

# 3.1.1 Coordinate Geometry

## Basic Coordinate Geometry

### What are cartesian coordinates?

- **Cartesian** coordinates are basically the *x*-*y* coordinate system
  - They allow us to label where things are in a two-dimensional plane
- In the 2D cartesian system, the horizontal axis is labelled *x* and the vertical axis is labelled *y*

### What can we do with coordinates?

- If we have two points with coordinates $(x_1, y_1)$ and $(x_2, y_2)$ then we should be able to find
  - The **midpoint** of the two points
  - The **distance** between the two points
  - The **gradient** of the line between them

### How do I find the midpoint of two points?

- The midpoint is the **average (middle) point**
  - It can be found by finding the middle of the *x*-coordinates and the middle of the *y*-coordinates
- The coordinates of the midpoint will be

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

  - This is given in the formula booklet under the prior learning section at the beginning

### How do I find the distance between two points?

- The distance between two points with coordinates $(x_1, y_1)$ and $(x_2, y_2)$ can be found using the formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

  - This is given in the formula booklet in the *prior learning* section at the beginning
- Pythagoras' Theorem $a^2 = b^2 + c^2$ is used to find the length of a line between two coordinates
- If the coordinates are labelled A and B then the line segment between them is written with the notation [AB]

### How do I find the gradient of the line between two points?

- The gradient of a line between two points with coordinates ($x_1$, $y_1$) and ($x_2$, $y_2$) can be found using the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

  - This is given in the formula booklet under section 2.1 Gradient formula
  - This is usually known as $m = \frac{\text{rise}}{\text{run}}$

## Worked example

Point *A* has coordinates (3, -4) and point *B* has coordinates (-5, 2).

i) Calculate the distance of the line segment *AB*.

A: (3, −4)  B: (−5, 2)

$x_1$  $y_1$  $x_2$  $y_2$

Formula for distance between two points:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

sub coordinates for A and B into the formula:

$$d = \sqrt{(3-(-5))^2 + (-4-2)^2}$$

$$= \sqrt{8^2 + (-6)^2} = \sqrt{100}$$

$$\boxed{d = 10 \text{ units}}$$

ii) Find the gradient of the line connecting points *A* and *B*.

A: (3, −4)  B: (−5, 2)

$x_1$ $y_1$ $x_2$ $y_2$

Formula for gradient of a line segment:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Sub coordinates for A and B into the formula:

$$m = \frac{2 - -4}{-5 - 3} = \frac{6}{-8} = -\frac{3}{4}$$

$$\boxed{m = -\frac{3}{4}}$$



iii) Find the midpoint of [*AB*] .

A: (3, −4)  B: (−5, 2)

$x_1$ $y_1$ $x_2$ $y_2$

Formula for the midpoint of two coordinates:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Sub values in:

$$\text{Midpoint} = \left(\frac{3 + (-5)}{2}, \frac{-4 + 2}{2}\right) = (-1, -1)$$

$$\boxed{\text{Midpoint} = (-1, -1)}$$

## Perpendicular Bisectors

### What is a perpendicular bisector?

- A perpendicular bisector of a line segment cuts the line segment in half at a right angle
  - Perpendicular lines meet at right angles
  - Bisect means to cut in half
- Two lines are perpendicular if the **product of their gradients is -1**

### How do I find the equation of the perpendicular bisector of a line segment?

- To find the equation of a straight line you need to find
  - The gradient of the line
  - A coordinate of a point on the line
- To find the equation of the **perpendicular bisector** of a line segment follow these steps:
  - STEP 1: Find the coordinates of the midpoint of the line segment
    - We know that the perpendicular bisector will cut the line segment in half so we can use the midpoint of the line segment as the known coordinate on the bisector
  - STEP 2: Find the gradient of the line segment
  - STEP 3: Find the gradient of the perpendicular bisector
    - This will be -1 divided by the gradient of the line segment
  - STEP 4: Substitute the gradient of the perpendicular bisector and the coordinates of the midpoint into an equation for a straight line
    - The **point-gradient** form $y - y_1 = m(x - x_1)$ is the easiest
  - STEP 5: Rearrange into the required form
    - Either $y = mx + c$ or $ax + by + d = 0$
    - These equations for a straight line are given in the formula booklet

## Worked example

Point $A$ has coordinates (4, -6) and point $B$ has coordinates (8, 6). Find the equation of the perpendicular bisector to [$AB$]. Give your answer in the form $ax + by + d = 0$.

Step 1: find the coordinates of the midpoint:

$A:(4, -6)$ $B:(8, 6)$

$x_1$ $y_1$ $x_2$ $y_2$

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Sub values in:

$$\text{Midpoint} = \left(\frac{4+8}{2}, \frac{-6+6}{2}\right) = (6, 0)$$

Step 2: Find the gradient of [AB]:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - -6}{8 - 4} = \frac{12}{4} = 3$$

Step 3: Find the gradient of the perpendicular bisector:

$$m_{\perp r} = -\frac{1}{m} = -\frac{1}{3}$$

Step 4: Substitute gradient and coordinate into an equation for a straight line.

insert coordinates of the midpoint.

$$(y - y_1) = m(x - x_1)$$
$$(y - 0) = -\frac{1}{3}(x - 6)$$

Step 5: Rearrange into the form $ax + by + d = 0$

$$(y - 0) = -\frac{1}{3}(x - 6) \quad (\times -3)$$
$$-3y = x - 6 \quad (+3y)$$

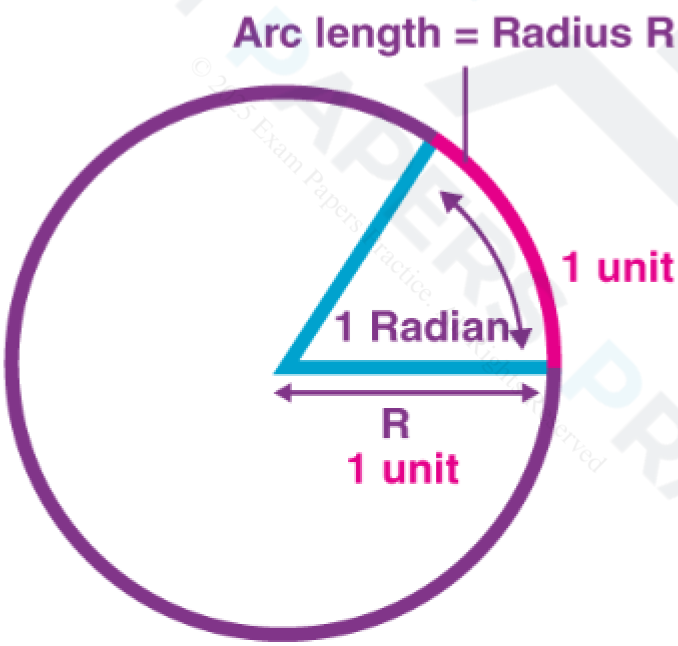
$$\boxed{x + 3y - 6 = 0}$$

## 3.1.2 Radian Measure

# Radian Measure

### What are radians?

- Radians are an alternative to degrees for measuring angles
- 1 radian is the angle in a **sector** of radius 1 and arc length 1
  - A circle with radius 1 is called a **unit circle**
- Radians are normally quoted in terms of $\pi$
  - $2\pi$ radians = 360°
  - $\pi$ radians = 180°
- The symbol for radians is $^c$ but it is more usual to see **rad**
  - Often, when $\pi$ is involved, no symbol is given as it is obvious it is in radians
  - Whilst it is okay to omit the symbol for radians, you should never omit the symbol for degrees
- In the exam you should use radians unless otherwise indicated




### How do I convert between radians and degrees?

- Use $\pi^c = 180°$ to convert between radians and degrees
  - To convert from radians to degrees multiply by $\dfrac{180}{\pi}$

- To convert from degrees to radians multiply by $\frac{\pi}{180}$

- Some of the common conversions are:
  - $2\pi^c = 360°$
  - $\pi^c = 180°$
  - $\frac{\pi}{2}^c = 90°$
  - $\frac{\pi}{3}^c = 60°$
  - $\frac{\pi}{4}^c = 45°$
  - $\frac{\pi}{6}^c = 30°$
- It is a good idea to remember some of these and use them to work out other conversions
- Your GDC will be able to work with both radians and degrees

## Worked example

i) Convert 43.8° to radians.

$$43.8^\circ \xrightarrow{\div 180^\circ} \frac{73}{300} \xrightarrow{\times \pi^c} \frac{73\pi}{300}^c$$

$(\pi^c = 180^\circ)$

$$\boxed{43.8^\circ = 0.764^c \text{ (3 s.f.)}}$$

ii) Convert $\frac{5\pi}{4}$ to degrees.

$$\frac{5\pi}{4} \xrightarrow{\div \pi^c} \frac{5}{4} \xrightarrow{\times 180^\circ} 225^\circ$$

$(\pi^c = 180^\circ)$

$$\boxed{\frac{5\pi}{4} = 225^\circ}$$

## 3.1.3 Arcs & Sectors

## Length of an Arc

### What is an arc?

- An arc is a part of the **circumference** of a circle
  - It is easiest to think of it as the crust of a single slice of pizza
- The length of an arc depends of the size of the angle at the centre of the circle
- If the angle at the centre is **less than 180°** then the arc is known as a **minor arc**
  - This could be considered as the crust of a single slice of pizza
- If the angle at the centre is **more than 180°** then the arc is known as a **major arc**
  - This could be considered as the crust of the remaining pizza after a slice has been taken away

### How do I find the length of an arc?

- The length of an arc is simply a fraction of the circumference of a circle
  - The fraction can be found by dividing the angle at the centre by 360°
- The formula for the length, $l$, of an arc is

$$l = \frac{\theta}{360} \times 2\pi r$$

  - Where $\theta$ is the angle measured in degrees
  - $r$ is the radius
  - This is **in the formula booklet**, you do not need to remember it

## Worked example

A circular pizza has had a slice cut from it, the angle of the slice that was cut was 38 °. The radius of the pizza is 12 cm. Find

i) the length of the outside crust of the slice of pizza (the minor arc),

A diagram will help:


Formula for the Length of an arc:

$$l = \frac{\theta}{360} \times 2\pi r$$

Substitute:

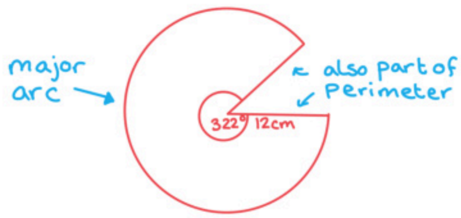
$$l = \frac{38}{360} \times 2\pi(12)$$

$$= \frac{38\pi}{15} = 7.9587\ldots \text{ cm}$$

$$\boxed{\text{Length of crust} = 7.96 \text{ cm (3s.f)}}$$

ii) the perimeter of the remaining pizza.

A diagram will help:




Formula for the Length of an arc:

$$l = \frac{\theta}{360} \times 2\pi r$$

Substitute:

$$l = \frac{322}{360} \times 2\pi(12)$$

$$= \frac{322\pi}{15} \leftarrow \text{Length of major arc}$$

Find perimeter:

$$P = \text{major arc} + \text{radius} + \text{radius}$$

$$= \frac{322\pi}{15} + 12 + 12 = 91.4395\ldots \text{ cm}$$

$$\boxed{\text{Perimeter} = 91.4 \text{ cm (3s.f)}}$$

# Area of a Sector

## What is a sector?

- A sector is a part of a circle enclosed by two radii (radiuses) and an arc
  - It is easier to think of this as the shape of a single slice of pizza
- The area of a sector depends of the size of the angle at the centre of the sector
- If the angle at the centre is **less than 180°** then the sector is known as a **minor sector**
  - This could be considered as the shape of a single slice of pizza
- If the angle at the centre is **more than 180°** then the sector is known as a **major sector**
  - This could be considered as the shape of the remaining pizza after a slice has been taken away

## How do I find the area of a sector?

- The area of a sector is simply a fraction of the area of the whole circle
  - The fraction can be found by dividing the angle at the centre by 360°
- The formula for the area, $A$, of a sector is

$$A = \frac{\theta}{360} \times \pi r^2$$

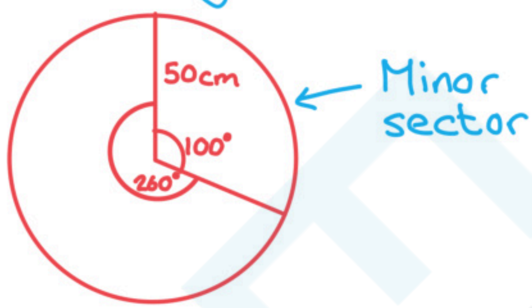
  - Where $\theta$ is the angle measured in degrees
  - $r$ is the radius
  - This is **in the formula booklet**, you do not need to remember it

### Worked example

Jamie has divided a circle of radius 50 cm into two sectors; a minor sector of angle 100° and a major sector of angle 260°. He is going to paint the minor sector blue and the major sector yellow. Find

i) the area Jamie will paint blue,

Start with a diagram:




Formula for the area of a sector:

$$A = \frac{\theta}{360°} \times \pi r^2$$

Substitute: $A = \frac{100}{360} \times \pi \times 50^2$

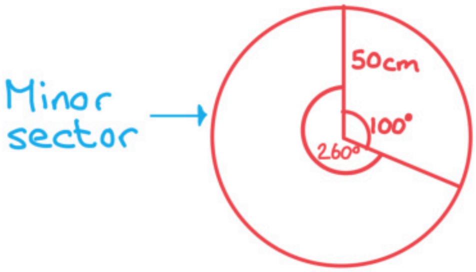
$$= \frac{6250}{9}\pi$$

$$= 2181.66\ldots \text{ cm}^2$$

$$\boxed{\text{Blue area} = 2180 \text{ cm}^2 \text{ (3sf)}}$$

ii) the area Jamie will paint yellow.

Start with a diagram:




Formula for the area of a sector:

$$A = \frac{\theta}{360°} \times \pi r^2$$

Substitute: $A = \frac{260}{360} \times \pi \times 50^2$

$$= \frac{16250}{9}\pi$$

$$= 5672.32\ldots \text{ cm}^2$$

**Yellow area = 5670 cm² (3sf)**

# Arcs & Sectors Using Radians

### How do I use radians to find the length of an arc?

- As the radian measure for a **full turn** is $2\pi$, the fraction of the circle becomes $\frac{\theta}{2\pi}$
- Working in radians, the formula for the length of an arc will become

$$l = \frac{\theta}{2\pi} \times 2\pi r$$

- Simplifying, the formula for the length, $l$, of an arc is

$$l = r\theta$$

  - $\theta$ is the angle measured in **radians**
  - $r$ is the radius
  - This is **given in the formula booklet,** you do not need to remember it

### How do I use radians to find the area of a sector?

- As the radian measure for a **full turn** is $2\pi$, the fraction of the circle becomes $\frac{\theta}{2\pi}$
- Working in radians, the formula for the area of a sector will become

$$A = \frac{\theta}{2\pi} \times \pi r^2$$

- Simplifying, the formula for the area, $A$, of a sector is

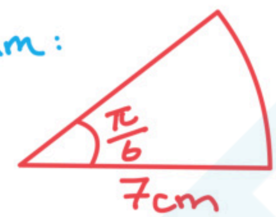
$$A = \frac{1}{2} r^2 \theta$$

  - $\theta$ is the angle measured in **radians**
  - $r$ is the radius
  - This is **given in the formula booklet,** you do not need to remember it

### Worked example

A slice of cake forms a sector of a circle with an angle of $\frac{\pi}{6}$ radians and radius of 7 cm. Find the area of the surface of the slice of cake and its perimeter.

Draw a diagram:




Area of a sector: $A = \frac{1}{2}r^2\theta$

Substitute: $r = 7, \ \theta = \frac{\pi}{6}$

$$A = \frac{1}{2}(7)^2\left(\frac{\pi}{6}\right) = \frac{49\pi}{12}$$

$$\boxed{\text{Area} = 12.8\text{ cm}^2 \ (3\text{ s.f.})}$$

Perimeter = arc Length + 2(radius)

Length of an arc: $l = r\theta$

$$P = 7\left(\frac{\pi}{6}\right) + 2(7)$$

$$\boxed{\text{Perimeter} = 17.7\text{ cm} \ (3\text{ s.f.})}$$