Oxford Cambridge and RSA

AS Level Further Mathematics B (MEI)

Y413/01 Modelling with Algorithms
Question Paper

Thursday 17 May 2018 – Afternoon
Time allowed: 1 hour 15 minutes

You must have:
- Printed Answer Booklet
- Formulae Further Mathematics B (MEI)

You may use:
- a scientific or graphical calculator

INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- Do **not** write in the barcodes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION
- The total number of marks for this paper is **60**.
- The marks for each question are shown in brackets **[ ]**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is used. You should communicate your method with correct reasoning.
- The Printed Answer Booklet consists of **12** pages. The Question Paper consists of **12** pages.

Answer **all** the questions.

1 Fig. 1 shows a network. The weights on the arcs are distances.

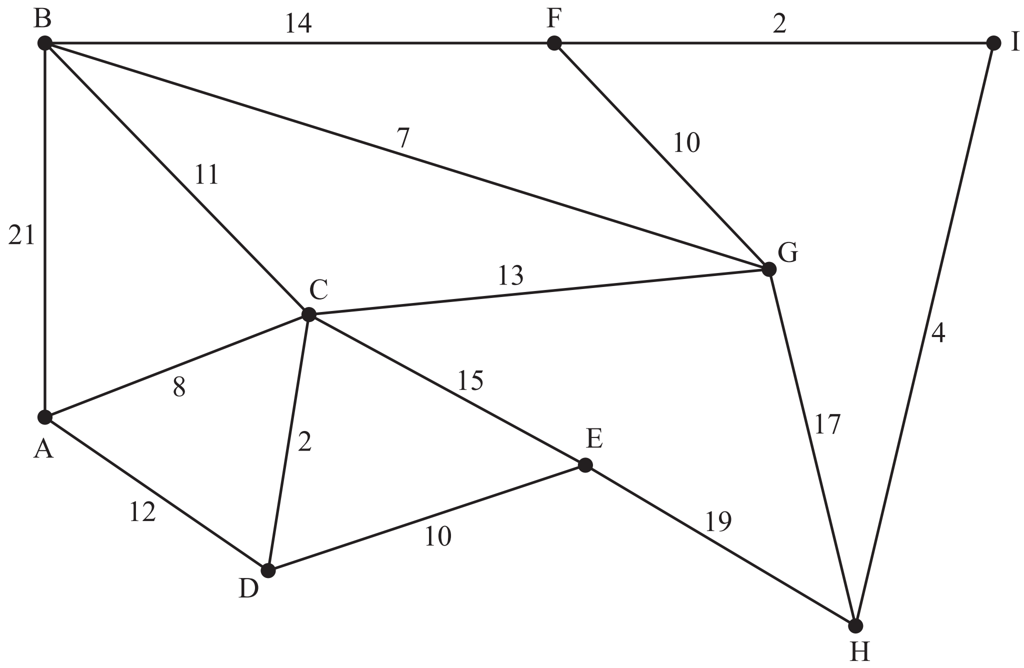

Fig. 1

(i) Apply Dijkstra's algorithm to the copy of the network in the Printed Answer Booklet to find the shortest path from A to H.

State each of the following.

- the shortest path
- its length

[6]

A problem requires a spanning tree that must include the arcs AD and GH.

(ii) The arcs AD and GH are chosen first. Apply Kruskal's algorithm to the ordered list of arcs in the Printed Answer Booklet to complete the tree. You should draw your tree and give its total length. **[4]**

(iii) How could you modify the network in Fig. 1 so that Prim's algorithm would find a spanning tree that includes arcs AD and GH? **[1]**

2 Fig. 2 represents a system of one-way roads through which traffic flows from a source, S, to a sink, T. The weights on the arcs show the capacities of the roads in cars per minute.

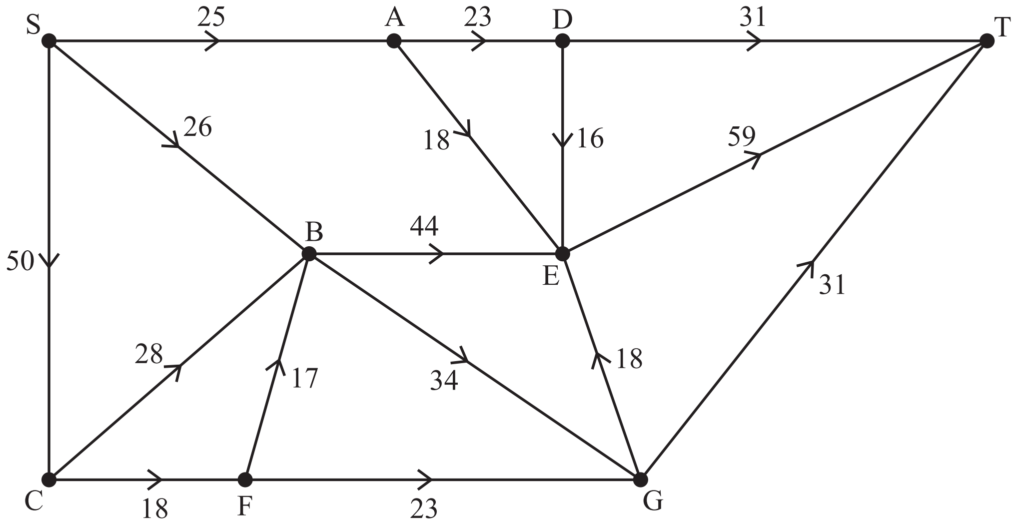

Fig. 2

(i) Explain why the maximum possible flow along FG must be less than 23 cars per minute. **[1]**

(ii) (A) The cut α partitions the vertices into the sets {S, C}, {A, B, D, E, F, G, T}. Calculate the capacity of cut α. **[1]**

(B) The cut β partitions the vertices into the sets {S, A, C, F, G}, {B, D, E, T}. Calculate the capacity of cut β. **[1]**

(iii) Given that one of α or β is a minimum cut, write down the maximum possible flow from S to T. **[1]**

On a particular morning

- road GE is closed for repairs,
- roads FB, AD and DE are full to capacity.

(iv) Use the diagram in the Printed Answer Booklet to show that the maximum possible flow from part **(iii)** is still achievable when GE is not used and FB, AD and DE are full to capacity. **[2]**

3 A list of n numbers is sorted by making passes through an algorithm as follows.

The first pass consists of the following.

> Compare the first number and second number. If necessary, swap them so that the first number is less than or equal to the second number.

The second pass consists of the following.

> Compare the second number and third number. If necessary, swap them so that the second number is less than or equal to the third number. If a swap occurred compare the new second number and the first number. If necessary, swap them so that the first number is less than or equal to the second number.

…

The $(n-1)$th (final) pass consists of the following.

> Compare the $(n-1)$th number and nth number. If necessary, swap them so that the $(n-1)$th number is less than or equal to the nth number. If a swap occurred compare the new $(n-1)$th number and the $(n-2)$th number. If necessary, swap them so that the $(n-2)$th number is less than or equal to the $(n-1)$th number. Continue in this way until no swap occurs between consecutive numbers or the first number has been swapped with the second number.

(i) Sort the following list of six numbers using the above algorithm

12 34 15 23 10 25

You should only show the result at the end of each pass. **[2]**

The algorithm is now applied to a list of n numbers.

(ii) Find, when $n = 6$,
- the minimum possible number of comparisons needed,
- the maximum possible number of comparisons needed.

[2]

The number of comparisons is used as a measure of the complexity of the above algorithm.

(iii) Determine, in the worst case, the complexity of the algorithm in terms of n. **[3]**

4 The table below shows the unit cost, in hundreds of pounds, of transporting goods from each of three suppliers, A, B and C to each of four depots, W, X, Y and Z. The margins of the table show the stock at each supplier and the demand at each depot.

	Demand	10	17	12	11
Stock		W	X	Y	Z
20	A	18	11	14	10
13	B	19	25	15	20
17	C	24	15	20	18

The following LP formulation can be used to find the minimum total cost of delivering all the required stock.

Minimise 18AW + 11AX + 14AY + 10AZ + 19BW + 25BX + 15BY + 20BZ + 24CW + 15CX + 20CY + 18CZ

subject to
AW + AX + AY + AZ = 20
BW + BX + BY + BZ = 13
CW + CX + CY + CZ = 17
AW + BW + CW = 10
AX + BX + CX = 17
AY + BY + CY = 12
AZ + BZ + CZ = 11

(i) Explain the purpose of each of the following lines from the LP formulation.

(*A*) Minimise 18AW + 11AX + … + 18CZ **[2]**

(*B*) AW + AX + AY + AZ = 20 **[1]**

The LP is run in an Online LP Solver and the following output is obtained.

OBJECTIVE FUNCTION VALUE

1) 726.0000

VARIABLE	VALUE	REDUCED COST
AW	9.000000	0.000000
AX	0.000000	0.000000
AY	0.000000	0.000000
AZ	11.000000	0.000000
BW	1.000000	0.000000
BX	0.000000	13.000000
BY	12.000000	0.000000
BZ	0.000000	9.000000
CW	0.000000	2.000000
CX	17.000000	0.000000
CY	0.000000	2.000000
CZ	0.000000	4.000000

The REDUCED COST value indicates by how much more the unit cost needs to be reduced for that particular transportation link to become part of the optimal solution. For example, the unit cost of transporting from B to Z would need to be reduced by at least £900 if BZ was to become part of the solution.

(ii) (*A*) Interpret the output to give a solution to the transportation problem. **[1]**

(*B*) State the minimum total cost of delivering all the required stock. **[1]**

It is later found that the unit cost of transporting goods from supplier B to depot X is only £1500

(iii) State the effect this change has on

(*A*) the objective function in the original LP formulation **[1]**

(*B*) the constraints in the original LP formulation. **[1]**

The modified LP problem (where the unit cost of transporting goods from B to X was reduced to £1500) is run in the same Online LP Solver and the following output is obtained.

OBJECTIVE FUNCTION VALUE

1) 726.0000

VARIABLE	VALUE	REDUCED COST
AW	9.000000	0.000000
AX	0.000000	0.000000
AY	0.000000	0.000000
AZ	11.000000	0.000000
BW	1.000000	0.000000
BX	0.000000	3.000000
BY	12.000000	0.000000
BZ	0.000000	9.000000
CW	0.000000	2.000000
CX	17.000000	0.000000
CY	0.000000	2.000000
CZ	0.000000	4.000000

In an attempt to cut the minimum total cost of delivering the required stock, supplier C decides to reduce the unit cost of supplying one of the four depots by £150. You can assume that the unit cost of transporting goods from B to X is still only £1500 and that no other supplier is going to cut their unit costs.

(iv) (*A*) Determine which of the four depots should have its transportation costs cut by supplier C to reduce the minimum total cost of delivery. Give a reason for your answer. **[2]**

(*B*) What is the new minimum total cost of delivering all the stock when supplier C makes this change? **[1]**

5 Fig. 5 shows an activity network for a project. Each activity is represented by an arc. The early event times and late event times are shown at each vertex.

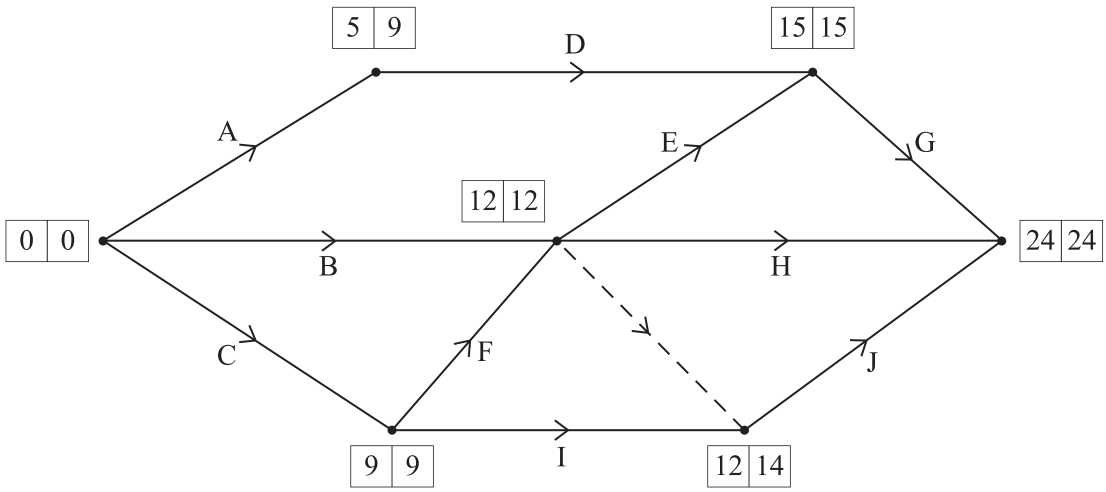

Fig. 5

(i) State the minimum completion time for the project. **[1]**

It is given that

- the project contains two critical paths that both include activity C,
- the total float for activity I is 3.

(ii) (*A*) Complete, as far as possible, the table in the Printed Answer Booklet showing the duration of each activity. Note that you will not be able to determine the duration for one of the activities. **[5]**

(*B*) Explain why one of the activity durations could not be determined in part **(ii)**(*A*). **[1]**

At the beginning of the project it becomes apparent that the duration of activity C can be reduced to 4 but this only reduces the minimum completion time of the project by 2.

(iii) Determine, showing your reasoning, the missing activity duration from part **(ii)**(*A*). **[2]**

Question 6 begins on page 10.

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6 The following LP problem is to be solved.

Maximise $P = 3x + y + 2z$

subject to
$$x + 2y \leqslant 30$$
$$2x + y + z \geqslant 14$$
$$2x + z \leqslant 20$$
$$x, y, z \geqslant 0$$

(i)

- Complete the initial tableau in the Printed Answer Booklet so that the two-stage simplex method may be used to solve this problem.
- Show how the constraints have been made into equations using slack variables, surplus variables and artificial variables.
- Show how the rows for the two objective functions are formed. **[7]**

After one iteration of the two-stage simplex method the tableau in Fig. 6.1 is produced.

Q	P	x	y	z	s_1	s_2	s_3	a_1	RHS
1	0	0	0	0	0	0	0	–1	0
0	1	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{3}{2}$	0	$\frac{3}{2}$	21
0	0	0	$\frac{3}{2}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	23
0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	7
0	0	0	–1	0	0	1	1	–1	6

Fig. 6.1

(ii) Explain how the tableau in Fig. 6.1 shows that the first stage has been completed. **[2]**

(iii) Using the tableau in Fig. 6.1 as the starting point for the second stage

- reduce the tableau so that the second stage can be started,
- carry out the first iteration of the second stage of the two-stage simplex method, using an entry in the s_2 column as the pivot element. **[3]**

(iv) State which variables are now basic after this first iteration of the second stage of the two-stage simplex method. **[1]**

The final tableau is shown in Fig. 6.2.

P	x	y	z	s_1	s_2	s_3	RHS
1	$\frac{3}{2}$	0	0	$\frac{1}{2}$	0	2	55
0	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	15
0	2	0	1	0	0	1	20
0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	1	1	21

Fig. 6.2

(v) Give the maximum value of P, and the corresponding values of x, y and z. **[2]**

(vi) The original LP problem is modified by replacing the constraint $2x+y+z \geqslant 14$ with $2x+y+z \geqslant k$. Determine the maximum possible value of k such that the solutions to both the modified and original LP problems would be the same. **[2]**

END OF QUESTION PAPER