DP IB Maths: AA HL

2.9 Further Functions & Graphs

Contents

2.9.1 Modulus Functions

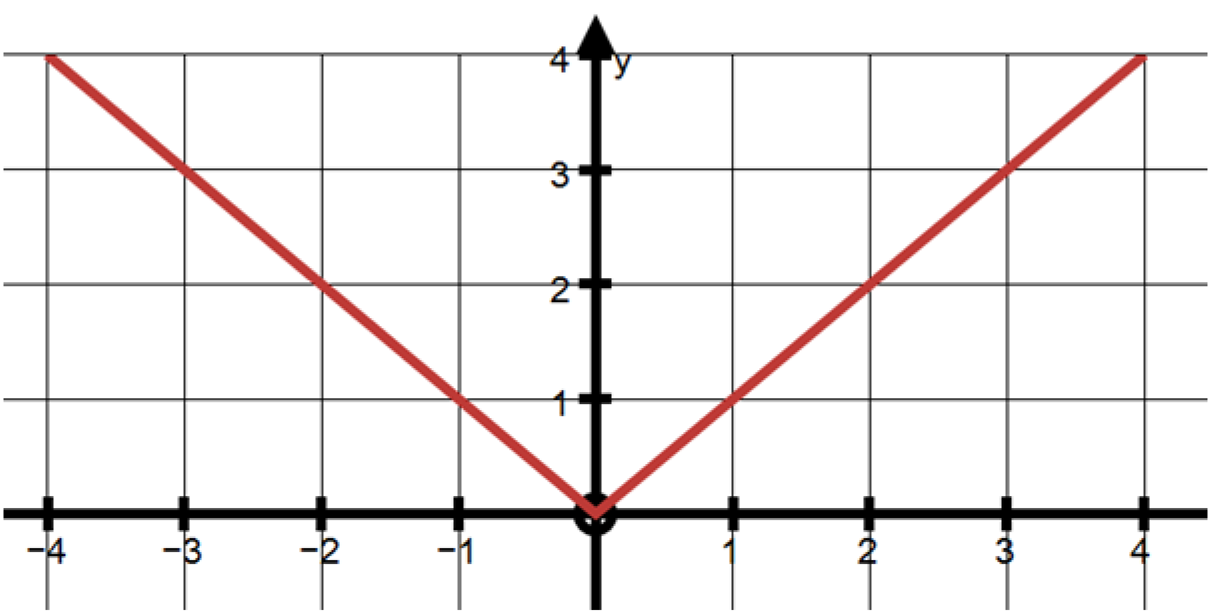
Modulus Functions & Graphs

What is the modulus function?

- The **modulus function** is defined by $f(x) = |x|$
  - $|x| = \sqrt{x^2}$
  - Equivalently it can be defined $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$
- Its **domain** is the set of **all real values**
- Its **range** is the set of **all real non-negative values**
- The modulus function gives the **distance** between 0 and *x*
  - This is also called the **absolute value** of *x*

What are the key features of the modulus graph: *y* = |*x*|?

- The graph has a **y-intercept** at (0, 0)
- The graph has **one root** at (0, 0)
- The graph has a **vertex** at (0, 0)
- The graph is **symmetrical** about the **y-axis**
- At the **origin**
  - The function is **continuous**
  - The function is **not differentiable**



What are the key features of the modulus graph: $y = a|x + p| + q$?

- Every **modulus graph** which is formed by **linear transformations** can be written in this form using key features of the modulus function
  - $|ax| = |a||x|$
    - For example: $|2x+1| = 2\left|x + \frac{1}{2}\right|$
  - $|p - x| = |x - p|$
    - For example: $|4 - x| = |x - 4|$
- The graph has a ***y*-intercept** when $x = 0$
- The graph can have 0, 1 or 2 **roots**
  - If a and q have the **same sign** then there will be **0 roots**
  - If $q = 0$ then there will be **1 root** at $(-p, 0)$
  - If a and q have **different signs** then there will be **2 roots** at $\left(-p \pm \frac{q}{a}, 0\right)$
- The graph has a **vertex** at $(-p, q)$
- The graph is **symmetrical** about the line $x = -p$
- The value of a determines the **shape** and the **steepness** of the graph
  - If a is **positive** the graph looks like ⋁
  - If a is **negative** the graph looks like ⋀
  - The **larger** the value of $|a|$ the **steeper** the lines
- At the **vertex**
  - The function is **continuous**
  - The function is **not differentiable**

2.9.2 Modulus Transformations

Modulus Transformations

How do I sketch the graph of the modulus of a function: $y = |f(x)|$?

- **STEP 1**: Keep the parts of the graph of $y = f(x)$ that are **on or above the x-axis**
- **STEP 2**: Any parts of the **graph below the x-axis** get **reflected** in the x-axis

How do I sketch the graph of a function of a modulus: $y = f(|x|)$?

- **STEP 1**: Keep the graph of $y = f(x)$ **only for $x \geq 0$**
- **STEP 2**: **Reflect** this in the **y-axis**

What is the difference between $y = |f(x)|$ and $y = f(|x|)$?

- The graph of $y = |f(x)|$ **never goes below the x-axis**
  - It does not have to have any lines of symmetry
- The graph of $y = f(|x|)$ is **always symmetrical about the x-axis**
  - It can go below the x-axis

When multiple transformations are involved how do I determine the order?

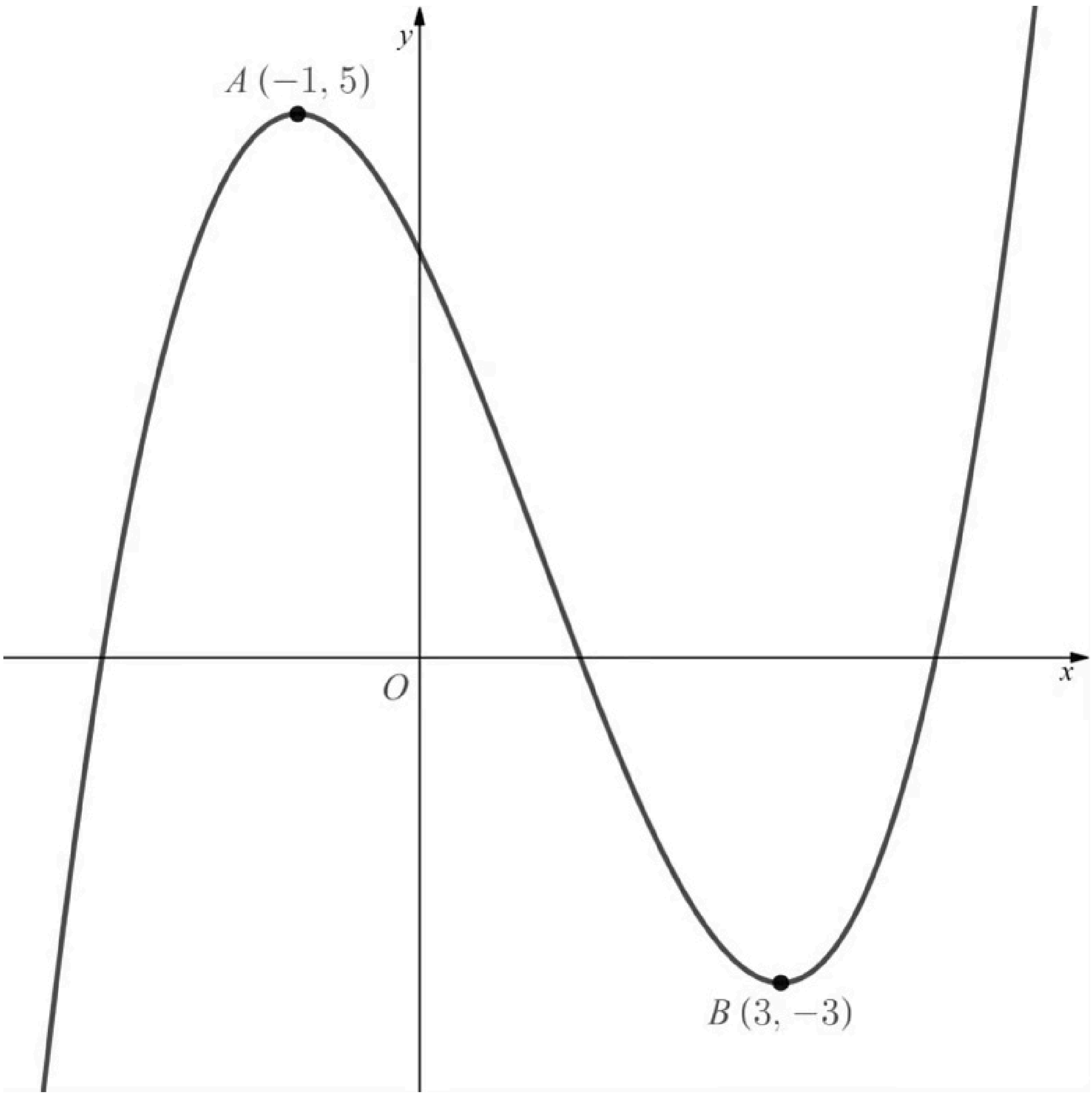
- The transformations **outside the function** follow the **same order** as the **order of operations**
  - $y = |af(x) + b|$
    - Deal with the a then the b then the modulus
  - $y = a|f(x)| + b$
    - Deal with the modulus then the a then the b
- The transformations **inside the function** are in the **reverse order** to the **order of operations**
  - $y = f(|ax + b|)$
    - Deal with the modulus then the b then the a
  - $y = f(a|x| + b)$
    - Deal with the b then the a then the modulus

Examiner Tip

- When sketching one of these transformations in an exam question make sure that the graphs do not look smooth at the points where the original graph have been reflected
  - For $y = |f(x)|$ the graph should look "sharp" at the points where it has been reflected on the x-axis
  - For $y = f(|x|)$ the graph should look "sharp" at the point where it has been reflected on the y-axis

Worked example

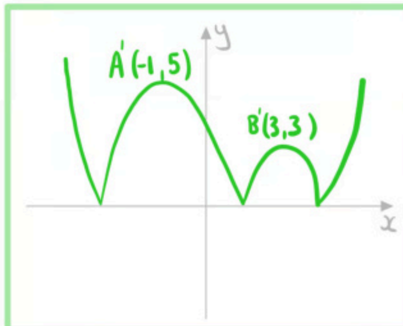
The diagram below shows the graph of $y = f(x)$.



(a) Sketch the graph of $y = |f(x)|$.

If the graph is on or above the x-axis then it stays the same
If the graph is below the x-axis the it is reflected in the x-axis

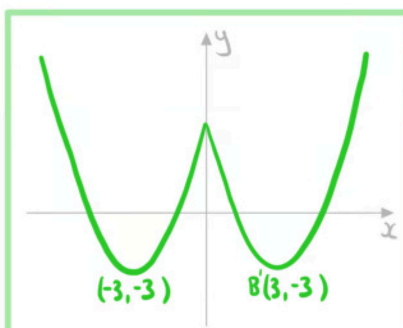
A stays the same (-1,5)
B becomes (3,3)



(b) Sketch the graph of $y = f(|x|)$.

keep the graph for $x \geq 0$
Reflect this in the y-axis

A disappears
B stays the same (3,-3)



2.9.3 Modulus Equations & Inequalities

Modulus Equations

How do I find the modulus of a function?

- The **modulus of a function *f*(*x*)** is
  - $|f(x)| = \begin{cases} f(x) & f(x) \geq 0 \\ -f(x) & f(x) < 0 \end{cases}$ or
  - $|f(x)| = \sqrt{[f(x)]^2}$

How do I solve modulus equations graphically?

- To solve $|f(x)| = g(x)$ graphically
  - Draw $y = |f(x)|$ and $y = g(x)$ into your GDC
  - Find the x-coordinates of the **points of intersection**

How do I solve modulus equations analytically?

- To solve $|f(x)| = g(x)$ analytically
  - Form **two equations**
    - $f(x) = g(x)$
    - $f(x) = -g(x)$
  - Solve both equations
  - **Check solutions** work in the original equation
    - For example: $x - 2 = 2x - 3$ has solution $x = 1$
    - But $|(1) - 2| = 1$ and $2(1) - 3 = -1$
    - So $x = 1$ is not a solution to $|x - 2| = 2x - 3$

Worked example

Solve for x:

a) $\left|\frac{2x+3}{2-x}\right| = 5$

Analytically

Split into two equations

$\frac{2x+3}{2-x} = \pm 5$

Solve individually

$\frac{2x+3}{2-x} = 5$

$2x+3 = 10-5x$

$7x = 7$

$x = 1$

$\frac{2x+3}{2-x} = -5$

$2x+3 = 5x-10$

$13 = 3x$

$x = \frac{13}{3}$

Check:

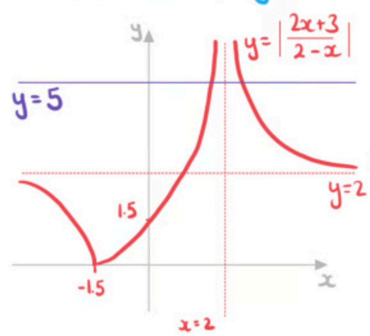
$\left|\frac{2(1)+3}{2-(1)}\right| = 5$ ✓

$\left|\frac{2\left(\frac{13}{3}\right)+3}{2-\left(\frac{13}{3}\right)}\right| = 5$ ✓

$x = 1$ or $x = \frac{13}{3}$

Graphically

Sketch the two graphs



Find the points of intersection

$(1, 5)$ $(4.33, 5)$

Choose the x-coordinates

$x = 1$ or $x = 4.33$ (3sf)

b) $|3x-1| = 5x-11$.

Analytically

Split into two equations

$3x - 1 = \pm(5x - 11)$

Solve individually

$3x - 1 = 5x - 11$	$3x - 1 = 11 - 5x$
$10 = 2x$	$8x = 12$
$x = 5$	$x = 1.5$

Check:

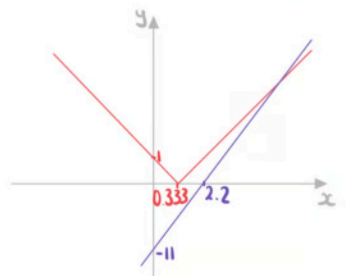
$|3(5) - 1| = 14$ ✓ $\quad |3(1.5) - 1| = 3.5$ ✗

$5(5) - 11 = 14 \quad 5(1.5) - 11 = -3.5$

$x = 5$

Graphically

Sketch the two graphs



Find the points of intersection

$(5, 14)$

Choose the x-coordinates

$x = 5$

Modulus Inequalities

How do I solve modulus inequalities analytically?

- To solve **any** modulus inequality
  - First solve the corresponding modulus equation
    - Remembering to **check whether solutions are valid**
  - Then use a graphical method or a sign table to find the intervals that satisfy the inequality
- Another method is to solve **two pairs of inequalities**
  - For $|f(x)| < g(x)$ solve:
    - $f(x) < g(x)$ when $f(x) \geq 0$
    - $f(x) > -g(x)$ when $f(x) \leq 0$
  - For $|f(x)| > g(x)$ solve:
    - $f(x) > g(x)$ when $f(x) \geq 0$
    - $f(x) < -g(x)$ when $f(x) \leq 0$

Worked example

Solve the following inequalities for x.

a) $|2x-1|<4$

Solve for $2x-1 \geq 0$

For $x \geq \frac{1}{2}$: $2x-1<4 \Rightarrow x<\frac{5}{2}$ $\therefore \frac{1}{2} \leq x < \frac{5}{2}$

Solve for $2x-1 \leq 0$

For $x \leq \frac{1}{2}$: $2x-1>-4 \Rightarrow x>-\frac{3}{2}$ $\therefore -\frac{3}{2} < x \leq \frac{1}{2}$

Combine inequalities $-\frac{3}{2} < x < \frac{5}{2}$

b) $|x+1|<|2x+3|$

Solve the corresponding equation

$|x+1| = |2x+3| \Rightarrow x+1 = \pm(2x+3)$

Solve $x+1 = 2x+3$ | $x+1 = -2x-3$

$x=-2$ | $x = -\frac{4}{3}$

Check $|(-2)+1| = 1$, $|2(-2)+3| = 1$ ✓ | $|(-\frac{4}{3})+1| = \frac{1}{3}$, $|2(-\frac{4}{3})+3| = \frac{1}{3}$ ✓

Use a sign table

Check $x=-3$	Check $x=-1.5$	Check $x=0$
$\lvert(-3)+1\rvert < \lvert 2(-3)+3\rvert$	$\lvert(-1.5)+1\rvert < \lvert 2(-1.5)+3\rvert$	$\lvert 0+1\rvert < \lvert 2(0)+3\rvert$
$2 < 3$	$0.5 < 0$	$1 < 3$
True ✓	False ✗	True ✓

-2 $\qquad -\frac{4}{3}$

Write solution $x<-2$ or $x>-\frac{4}{3}$

2.9.4 Reciprocal & Square Transformations

Reciprocal Transformations

What effects do reciprocal transformations have on the graphs?

- The ***x*-coordinates stay the same**
- The ***y*-coordinates change**
  - Their values become their **reciprocals**
- The coordinates (*x*, *y*) become $\left(x, \frac{1}{y}\right)$ where *y* ≠ 0
  - If *y* = 0 then a vertical asymptote goes through the original coordinate
  - Points that lie on the line ***y* = 1** or the line ***y* = -1 stay the same**

How do I sketch the graph of the reciprocal of a function: *y* = 1/*f*(*x*)?

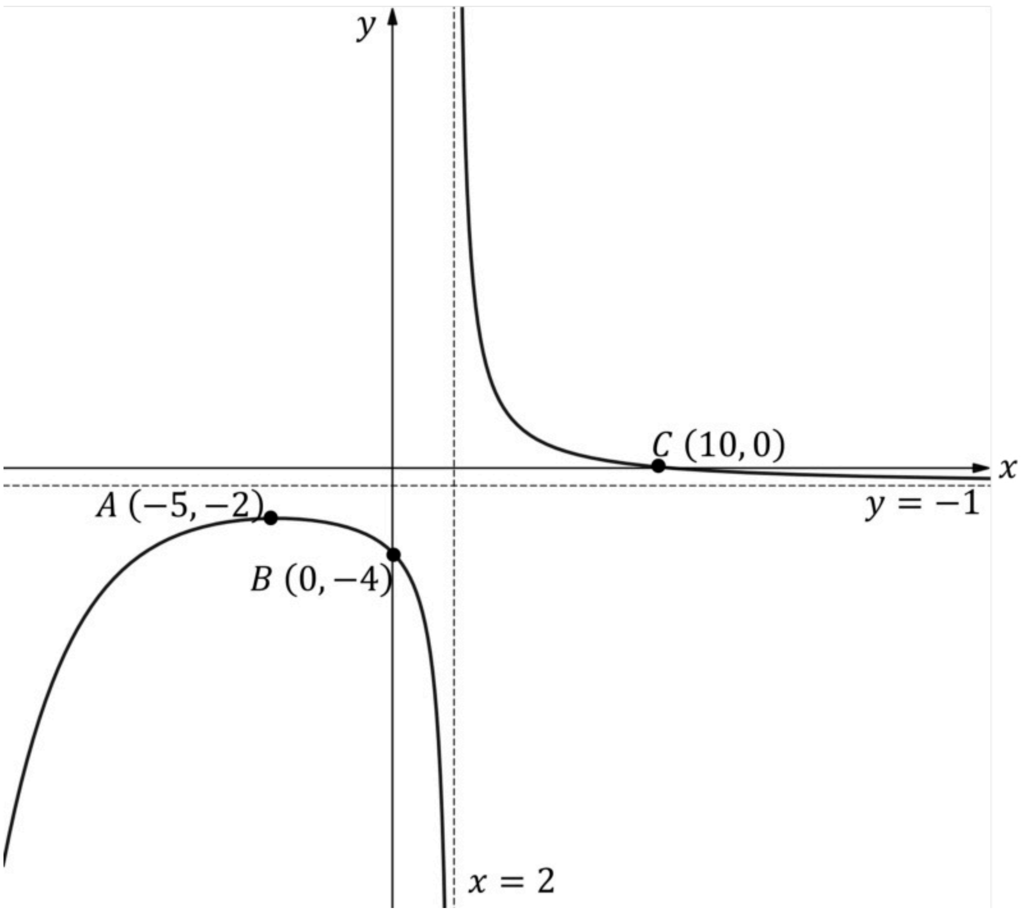
- Sketch the **reciprocal transformation** by considering the **different features** of the original graph
- Consider key points on the original graph
  - If (x_1, y_1) is a point on *y* = *f*(*x*) where $y_1 \neq 0$
    - $\left(x_1, \frac{1}{y_1}\right)$ is a point on $y = \frac{1}{f(x)}$
    - If $|y_1| < 1$ **then the point gets further away from the *x*-axis**
    - If $|y_1| > 1$ **then the point gets closer to the *x*-axis**
  - If *y* = *f*(*x*) has a ***y*-intercept** at (0, c) where c ≠ 0
    - The reciprocal graph $y = \frac{1}{f(x)}$ has a ***y*-intercept** at $\left(0, \frac{1}{c}\right)$
  - If *y* = *f*(*x*) has a **root** at (*a*, 0)
    - The reciprocal graph $y = \frac{1}{f(x)}$ has a **vertical asymptote** at $x = a$
  - If *y* = *f*(*x*) has a **vertical asymptote** at $x = a$
    - The reciprocal graph $y = \frac{1}{f(x)}$ has a **discontinuity** at (*a*, 0)
    - The **discontinuity** will look like a **root**

- If $y = f(x)$ has a **local maximum** at (x_1, y_1) where $y_1 \neq 0$
  - The reciprocal graph $y = \frac{1}{f(x)}$ has a **local minimum** at $\left(x_1, \frac{1}{y_1}\right)$
- If $y = f(x)$ has a **local minimum** at (x_1, y_1) where $y_1 \neq 0$
  - The reciprocal graph $y = \frac{1}{f(x)}$ has a **local maximum** at $\left(x_1, \frac{1}{y_1}\right)$

- Consider key regions on the original graph
  - If $y = f(x)$ is **positive** then $y = \frac{1}{f(x)}$ is **positive**
    - If $y = f(x)$ is **negative** then $y = \frac{1}{f(x)}$ is **negative**
  - If $y = f(x)$ is **increasing** then $y = \frac{1}{f(x)}$ is **decreasing**
    - If $y = f(x)$ is **decreasing** then $y = \frac{1}{f(x)}$ is **increasing**
  - If $y = f(x)$ has a **horizontal asymptote** at $y = k$
    - $y = \frac{1}{f(x)}$ has a **horizontal asymptote** at $y = \frac{1}{k}$ if $k \neq 0$
    - $y = \frac{1}{f(x)}$ **tends to** $\pm\infty$ if $k = 0$
  - If $y = f(x)$ **tends to** $\pm\infty$ as x tends to $+\infty$ or $-\infty$
    - $y = \frac{1}{f(x)}$ has a **horizontal asymptote** at $y = 0$

Worked example

The diagram below shows the graph of $y = f(x)$ which has a local maximum at the point *A*.



Sketch the graph of $y = \frac{1}{f(x)}$.

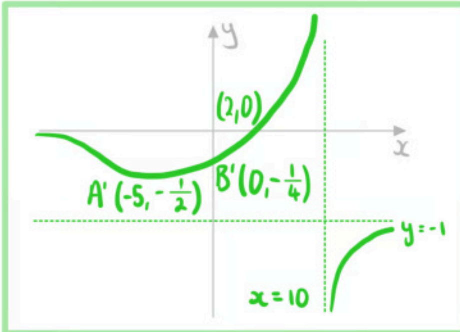
A becomes local minimum $(-5, -\frac{1}{2})$

Vertical asymptote becomes root $(2, 0)$

B becomes $(0, -\frac{1}{4})$

C becomes vertical asymptote $x = 10$

Horizontal asymptote $y = -1$ remains



Square Transformations

What effects do square transformations have on the graphs?

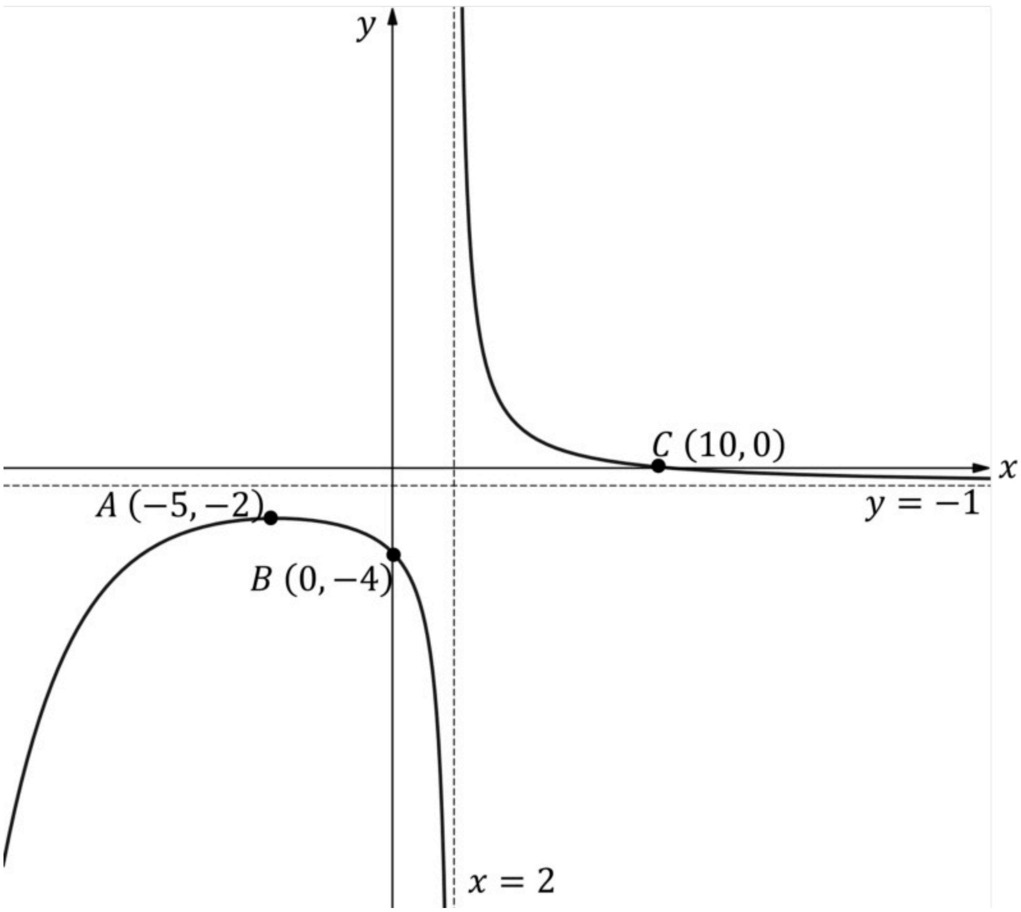
- The effects are **similar to** the transformation ***y* = |*f*(*x*)|**
  - The parts **below the *x*-axis are reflected**
  - The **vertical distance** between a point and the *x*-axis is **squared**
    - This has the effect of **smoothing the curve** at the *x*-axis
- $y=[f(x)]^2$ is **never below the *x*-axis**
- The ***x*-coordinates stay the same**
- The ***y*-coordinates change**
  - Their values are **squared**
- The coordinates (x, y) become (x, y^2)
  - Points that lie on the ***x*-axis** or the line ***y* = 1 stay the same**

How do I sketch the graph of the square of a function: y = [*f*(*x*)]²?

- Sketch the **square transformation** by considering the **different features** of the original graph
- Consider key points on the original graph
  - If (x_1, y_1) is a point on $y = f(x)$
    - (x_1, y_1^2) is a point on $y=[f(x)]^2$
    - If $|y_1| < 1$ then the point gets **closer to the *x*-axis**
    - If $|y_1| > 1$ then the point gets **further away from the *x*-axis**
  - If $y = f(x)$ has a **y-intercept** at (0, c)
    - The square graph $y=[f(x)]^2$ has a **y-intercept** at $(0, c^2)$
  - If $y = f(x)$ has a **root** at $(a, 0)$
    - The square graph $y=[f(x)]^2$ has a **root** and **turning point** at $(a, 0)$
  - If $y = f(x)$ has a **vertical asymptote** at $x = a$
    - The square graph $y=[f(x)]^2$ has a **vertical asymptote** at $x = a$
  - If $y = f(x)$ has a **local maximum** at (x_1, y_1)
    - The square graph $y=[f(x)]^2$ has a **local maximum** at (x_1, y_1^2) if $y_1 > 0$
    - The square graph $y=[f(x)]^2$ has a **local minimum** at (x_1, y_1^2) if $y_1 \leq 0$
  - If $y = f(x)$ has a **local minimum** at (x_1, y_1)
    - The square graph $y=[f(x)]^2$ has a **local minimum** at (x_1, y_1^2) if $y_1 \geq 0$
    - The square graph $y=[f(x)]^2$ has a **local maximum** at (x_1, y_1^2) if $y_1 < 0$

Worked example

The diagram below shows the graph of $y = f(x)$ which has a local maximum at the point A.



Sketch the graph of $y = [f(x)]^2$.

A becomes local minimum (-5, 4)

Vertical asymptote x = 2 remains

B becomes (0, 16)

C becomes local minimum

Horizontal asymptote becomes y = 1