DP IB Maths: AA HL

2.8 Inequalities

Contents

2.8.1 Solving Inequalities Graphically

Solving Inequalities Graphically

How can I solve inequalities graphically?

- Consider the inequality $f(x) \le g(x)$, where $f(x)$ and $g(x)$ are functions of x
  - if we **move $g(x)$ to the LHS** we get
    - $f(x) - g(x) \le 0$
- Solve **$f(x) - g(x) = 0$ to** find the **zeros** of $f(x) - g(x)$
  - These correspond to the x-coordinates of the points of intersection of the graphs $y = f(x)$ and $y = g(x)$
- To solve the inequality we can use a **graph**
  - **Graph $y = f(x) - g(x)$** and label its zeros
  - Hence find the intervals of x that satisfy the inequality $f(x) - g(x) \le 0$
    - These are the **intervals which satisfies the original inequality** $f(x) \le g(x)$
  - This method is particularly useful when finding the intersections between the functions is difficult due to needing large x and y windows on your GDC

Be careful when rearranging inequalities!

- Remember to **flip the sign** of the inequality when you **multiply or divide** both sides by a **negative** number
  - e. **1 < 2** → [times both sides by (–1)] → **–1 > –2** (sign flips)
- **Never multiply or divide** by a **variable** as this could be **positive or negative**
  - You can only multiply by a term if you are certain it is always positive (or always negative)
    - Such as $x^2,\ |x|,\ \mathrm{e}^x$
- Some **functions reverse the inequality**
  - Taking reciprocals of positive values
    - $0 < x < y \Rightarrow \dfrac{1}{x} > \dfrac{1}{y}$
  - Taking logarithms when the base is $0 < a < 1$
    - $0 < x < y \Rightarrow \log_a(x) > \log_a(y)$
- The **safest way** to rearrange is simply to add & subtract to move all the terms onto one side

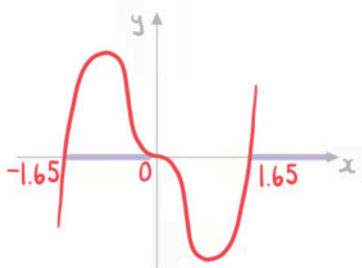
Worked example

Use a GDC to solve the inequality $2x^3 < x^5 - 2x$.

Rearrange to get one side as zero

$x^5 - 2x^3 - 2x > 0$

On GDC sketch $y = x^5 - 2x^3 - 2x$ and find zeros



Identify the sections where the graph is above the x-axis

$\boxed{-1.65 < x < 0 \quad \text{or} \quad x > 1.65}$

2.8.2 Polynomial Inequalities

Polynomial Inequalities

How do I solve polynomial inequalities?

- **STEP 1**: **Rearrange the inequality** so that **one of the sides is equal to zero**
  - For example: $P(x) \le 0$
- **STEP 2**: Find the **roots** of the polynomial
  - You can do this by factorising or using GDC to solve $P(x) = 0$
- **STEP 3**: Choose one of the following methods:
- **Graph method**
  - Sketch a graph of the polynomial (with or without a GDC)
  - Choose the intervals for x corresponding to the sections of the graph that satisfy the inequality
    - For example: for $P(x) \le 0$ you would want the sections below the x-axis
- **Sign table method**
  - If you are unsure how to sketch a polynomial graph then this method is best
  - **Split the real numbers** into the possible **intervals** using the roots
    - If the roots are a and b then the intervals would be $x<a$, $a<x<b$, $x>b$
  - **Test a value** from each interval using the inequality
    - Choose a value within an interval and substitute into $P(x)$ to determine if it is positive or negative
  - Alternatively if the polynomial is factorised you can **determine the sign of each factor** in each interval
    - An odd number of negative factors in an interval will mean the polynomial is negative on that interval
  - If the value satisfies the inequality then that interval is part of the solution

Worked example

Solve the inequality $x^3 + 2x^2 > x + 2$ using an algebraic method.

Rearrange $x^3 + 2x^2 - x - 2 > 0$

Let $P(x) = x^3 + 2x^2 - x - 2$

Find a factor $P(1) = 0 \quad \therefore \quad (x-1)$ is a factor

Factorise $(x-1)(x^2+3x+2) > 0$ Compare coefficients or use division

$(x-1)(x+1)(x+2) > 0$

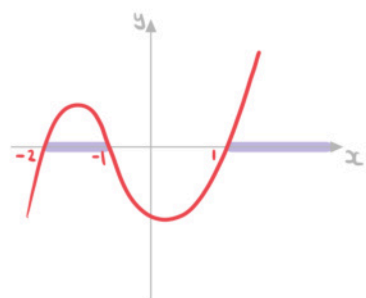
Find the roots $1, -1, -2$

Construct a sign table

For $x < -2$:	For $-2 < x < -1$:	For $-1 < x < 1$:	For $x > 1$:
$(x+2) < 0$	$(x+2) > 0$	$(x+2) > 0$	$(x+2) > 0$
$(x+1) < 0$	$(x+1) < 0$	$(x+1) > 0$	$(x+1) > 0$
$(x-1) < 0$	$(x-1) < 0$	$(x-1) < 0$	$(x-1) > 0$
$\therefore P(x) < 0$	$\therefore P(x) > 0$	$\therefore P(x) < 0$	$\therefore P(x) > 0$
✗	✓	✗	✓

$-2 \qquad -1 \qquad 1$

Or sketch



Choose the regions that satisfy the inequality

$$\boxed{-2 < x < -1 \quad \text{or} \quad x > 1}$$