DP IB Maths: AA HL

2.7 Polynomial Functions

Contents

2.7.1 Factor & Remainder Theorem

Factor Theorem

What is the factor theorem?

- The **factor theorem** is used to find the linear factors of **polynomial** equations
- This topic is closely tied to finding the **zeros** and **roots** of a **polynomial** function/equation
  - As a rule of thumb a **zero** refers to the polynomial function and a **root** refers to a polynomial equation
- For any **polynomial** function $P(x)$
  - $(x - k)$ is a **factor** of $P(x)$ if $\mathbf{P(k) = 0}$
  - $\mathbf{P(k) = 0}$ if $(x - k)$ is a **factor** of $P(x)$

How do I use the factor theorem?

- Consider the polynomial function $P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ and $(x - k)$ is a **factor**
  - Then, due to the factor theorem $P(k) = a_n k^n + a_{n-1}k^{n-1} + \dots + a_1 k + a_0 = 0$
  - $P(x) = (x - k) \times Q(x)$, where $Q(x)$ is a **polynomial** that is a factor of $P(x)$
  - Hence, $\dfrac{P(x)}{x - k} = Q(x)$, where $Q(x)$ is another factor of $P(x)$
- If the linear factor has a **coefficient of *x*** then you must first factorise out the coefficient
  - If the linear factor is $(ax - b) = a\left(x - \dfrac{b}{a}\right) \rightarrow P\left(\dfrac{b}{a}\right) = 0$

Worked example

Determine whether $(x-2)$ is a factor of the following polynomials:

a) $f(x) = x^3 - 2x^2 - x + 2$.

Step 1: Determine k

Our linear function is $x - 2$

$\rightarrow$ so $k = 2$

Step 2: Apply factor theorem

For $x - 2$ to be a factor of $f(x)$,

$f(2)$ has to equal zero

$f(2) = (2)^3 - 2(2)^2 - (2) + 2$

$= 8 - 8 - 2 + 2$

$= 0$

$f(2) = 0$,

so $x - 2$ is a factor of $f(x)$

b) $g(x) = 2x^3 + 3x^2 - x + 5$.

Step 1: Determine k

Our linear function is $x - 2$

$\rightarrow$ so $k = 2$

Step 2: Apply factor theorem

For $x - 2$ to be a factor of $g(x)$,
$g(2)$ has to equal zero

$$g(2) = 2(2)^3 + 3(2)^2 - (2) + 5$$
$$= 16 - 12 - 2 + 5$$
$$= 7$$

$g(2) = 7$,
so $x - 2$ is not a factor of $g(x)$

It is given that $(2x - 3)$ is a factor of $h(x) = 2x^3 - bx^2 + 7x - 6$.

c) Find the value of b.

Step 1: Determine k

Our linear function is $2x - 3$

$\rightarrow$ so $k = \frac{3}{2}$

Step 2: Apply factor theorem to find b

Since $2x-3$ is a factor of $h(x)$,

$h\left(\frac{3}{2}\right) = 0$

$$0 = 2\left(\frac{3}{2}\right)^3 - b\left(\frac{3}{2}\right)^2 + 7\left(\frac{3}{2}\right) - 6$$

$$= \frac{54}{8} - \frac{9}{4}b + \frac{21}{2} - 6$$

$\boxed{b = 5}$

Remainder Theorem

What is the remainder theorem?

- The **remainder theorem** is used to find the remainder when we divide a **polynomial** function by a linear function
- When any polynomial $P(x)$ is divided by any linear function $(x - k)$ the value of the remainder R is given by $P(k) = R$
  - Note, when $P(k) = 0$ then $(x - k)$ is a factor of $P(x)$

How do I use the remainder theorem?

- Consider the polynomial function $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ and the linear function $(x - k)$
  - Then, due to the remainder theorem $P(k) = a_n k^n + a_{n-1}k^{n-1} + \ldots + a_1 k + a_0 = R$
  - $P(x) = (x - k) \times Q(x) + R$, where $Q(x)$ is a **polynomial**
  - Hence, $\dfrac{P(x)}{x-k} = Q(x) + \dfrac{R}{x-k}$, where R is the remainder
- If the linear function has a **coefficient of *x*** then you must first factorise out the coefficient
  - If the linear function is $(ax - b) = a\left(x - \dfrac{b}{a}\right) \to P\left(\dfrac{b}{a}\right) = R$

Worked example

Let $f(x) = 2x^4 - 2x^3 - x^2 - 3x + 1$, find the remainder R when $f(x)$ is divided by:

a) $x - 3$.

Step 1: Determine k

Our linear function is $x - 3$

$\rightarrow$ so $k = 3$

Step 2: Apply remainder theorem

$f(3) = R$

$f(3) = 2(3)^4 - 2(3)^3 - (3)^2 - 3(3) + 1$

$f(3) = 162 - 54 - 9 - 9 + 1$

$f(3) = 91$

$\boxed{R = 91}$

b) $x + 2$.

Step 1: Determine k

Our linear function is $x + 2$

$\rightarrow$ so $k = -2$

Step 2: Apply remainder theorem

$f(-2) = R$

$f(-2) = 2(-2)^4 - 2(-2)^3 - (-2)^2 - 3(-2) + 1$

$f(-2) = 32 + 16 - 4 + 6 + 1$

$f(-2) = 51$

$\boxed{R = 51}$

The remainder when $f(x)$ is divided by $(2x + k)$ is $\frac{893}{8}$.

c) Given that $k > 0$, find the value of k.

Step 1: Apply remainder theorem

$2x + k = 2\left(x + \frac{k}{2}\right)$ $\quad f\left(-\frac{k}{2}\right) = \frac{893}{8}$

$\frac{893}{8} = 2\left(-\frac{k}{2}\right)^4 - 2\left(-\frac{k}{2}\right)^3 - \left(-\frac{k}{2}\right)^2 - 3\left(-\frac{k}{2}\right) + 1$

Step 2: Solve for k using your GDC

$\boxed{k = 5}$

2.7.2 Polynomial Division

Polynomial Division

What is polynomial division?

- Polynomial division is the process of **dividing two polynomials**
  - This is usually only useful when the **degree of the denominator** is **less than or equal** to the **degree of the numerator**
- To do this we use an algorithm similar to that used for **division of integers**
- To divide the polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ by the polynomial $D(x) = b_k x^k + b_{k-1} x^{k-1} + \ldots + b_1 x + b_0$ where $k \le n$
  - STEP 1
    Divide the **leading term of the polynomial** $P(x)$ by **the leading term of the divisor** $D(x)$:
    $$\frac{a_n x^n}{b_b x^k} = q_m x^m$$
  - STEP 2
    Multiply the divisor by this term: $D(x) \times q_m x^m$
  - STEP 3
    Subtract this from the **original polynomial** $P(x)$ to cancel out the leading term:
    $R(x) = P(x) - D(x) \times q_m x^m$
  - Repeat steps 1 – 3 using the new polynomial $R(x)$ in place of $P(x)$ until the subtraction results in an expression for $R(x)$ with degree less than the divisor
    - The quotient $Q(x)$ is the **sum of the terms** you multiplied the divisor by:
      $Q(x) = q_m x^m + q_{m-1} x^{m-1} + \ldots + q_1 x + q_0$
    - The remainder $R(x)$ is the polynomial after the final subtraction

Division by linear functions

- If $P(x)$ has degree n and is divided by a linear function $(ax + b)$ then
  - $\frac{P(x)}{ax + b} = Q(x) + \frac{R}{ax + b}$ where
    - $ax + b$ is the **divisor** (degree 1)
    - $Q(x)$ is the **quotient** (degree n – 1)
    - R is the **remainder** (degree 0)
  - Note that $P(x) = Q(x) \times (ax + b) + R$

Division by quadratic functions

- If $P(x)$ has degree n and is divided by a quadratic function $(ax^2 + bx + c)$ then
  - $\dfrac{P(x)}{ax^2 + bx + c} = Q(x) + \dfrac{ex + f}{ax^2 + bx + c}$ where
    - $ax^2 + bx + c$ is the **divisor** (degree 2)
    - $Q(x)$ is the **quotient** (degree $n - 2$)
    - $ex + f$ is the **remainder** (degree less than 2)
  - The remainder will be **linear** (degree 1) if $e \neq 0$, and **constant** (degree 0) if $e = 0$
  - Note that $P(x) = Q(x) \times (ax^2 + bx + c) + ex + f$

Division by polynomials of degree *k* ≤ *n*

- If $P(x)$ has degree n and is divided by a polynomial $D(x)$ with degree $k \leq n$
  - $\dfrac{P(x)}{D(x)} = Q(x) + \dfrac{R(x)}{D(x)}$ where
    - $D(x)$ is the **divisor** (degree k)
    - $Q(x)$ is the **quotient** (degree $n - k$)
    - $R(x)$ is the **remainder** (degree less than k)
  - Note that $P(x) = Q(x) \times D(x) + R(x)$

Are there other methods for dividing polynomials?

- **Synthetic division** is a faster and shorter way of setting out a division when dividing by a linear term of the form
  - To divide $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ by $(x - c)$:
    - Set $b_n = a_n$
    - Calculate $b_{n-1} = a_{n-1} + c \times b_n$
    - Continue this iterative process $b_{i-1} = a_{i-1} + c \times a_i$
    - The quotient is $Q(x) = b_n x^{n-1} + b_{n-1} x^{n-2} + \ldots + b_2 x + b_1$ and the remainder is $r = b_0$
- You can also find quotients and remainders by **comparing coefficients**
  - Given a polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$
  - And a divisor $D(x) = d_k x^k + d_{k-1} x^{k-1} + \ldots + d_1 x + d_0$
  - Write $Q(x) = q_{n-k} x^{n-k} + \ldots + q_1 x + q_0$ and $R(x) = r_{k-1} x^{k-1} + \ldots + r_1 x + r_0$
  - Write $P(x) = Q(x)D(x) + R(x)$
    - Expand the right-hand side
    - Equate the coefficients
    - Solve to find the unknowns q's & r's

Worked example

a) Perform the division $\frac{x^4+11x^2-1}{x+3}$. Hence write x^4+11x^2-1 in the form $Q(x)\times(x+3)+R$.

Step 1: what do we multiply x by to get x^4?

$$\begin{array}{r|l} & x^3 \\ \hline x+3 & x^4+0x^3+11x^2+0x-1 \end{array}$$

Note: $0x^3$ and $0x$ are used to keep like terms together.

Step 2: subtract $x^3(x+3)=x^4+3x^3$ from x^4+0x^3

$$\begin{array}{r|l} & x^3 \\ \hline x+3 & x^4+0x^3+11x^2+0x-1 \\ & -(x^4+3x^3) \\ \hline & \quad\;\; -3x^3 \end{array}$$

Step 3: bring the $11x^2$ down and return to step 1.

$$
\begin{array}{r|l}
 & x^3 - 3x^2 + 20x - 60 \\
\hline
x+3 & x^4 + 0x^3 + 11x^2 + 0x - 1 \\
 & -(x^4 + 3x^3) \\
\hline
 & -3x^3 + 11x^2 \\
 & -(-3x^3 - 9x^2 \\
\hline
 & 20x^2 + 0x \\
 & -(20x^2 + 60x) \\
\hline
 & -60x - 1 \\
 & -(-60x - 180) \\
\hline
 & 179
\end{array}
$$

$$\boxed{\begin{aligned} &x^4 + 11x^2 - 1 \\ &= (x^3 - 3x^2 + 20x - 60)(x+3) + 179 \end{aligned}}$$

b) Find the quotient and remainder for $\dfrac{x^4 + 4x^3 - x + 1}{x^2 - 2x}$. Hence write $x^4 + 4x^3 - x + 1$ in the form $Q(x) \times (x^2 - 2x) + R(x)$.

When dividing by quadratics use the same steps as above.

$$
\begin{array}{r|l}
 & \quad\quad\quad\quad\quad\quad x^2 + 6x + 12 \\
\hline
x^2 - 2x & x^4 + 4x^3 + 0x^2 - x + 1 \\
 & -\underline{(x^4 - 2x^3)} \\
 & \quad\quad 6x^3 + 0x^2 \\
 & \quad -\underline{(6x^3 - 12x^2)} \\
 & \quad\quad\quad 12x^2 - x \\
 & \quad\quad -\underline{(12x^2 - 24x)} \\
 & \quad\quad\quad\quad 23x + 1
\end{array}
$$

$$\boxed{\begin{aligned} &x^4 + 4x^3 - x + 1 \\ &= (x^2 + 6x + 12)(x^2 - 2x) + 23x + 1 \end{aligned}}$$

2.7.3 Polynomial Functions

Sketching Polynomial Graphs

In exams you'll commonly be asked to sketch the graphs of different polynomial functions with and without the use of your GDC.

What's the relationship between a polynomial's degree and its zeros?

- If a **real polynomial** $P(x)$ has **degree *n***, it will have ***n* zeros** which can be written in the form $a + bi$, where $a, b \in \mathbb{R}$
  - For example:
    - A quadratic will have 2 zeros
    - A cubic function will have 3 zeros
    - A quartic will have 4 zeros
  - Some of the zeros may be **repeated**
- Every **real polynomial** of **odd degree** has **at least one real zero**

How do I sketch the graph of a polynomial function without a GDC?

- Suppose $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ ... + a_1 x + a_0$ is a **real polynomial** with **degree *n***
- To sketch the graph of a polynomial you need to know three things:
  - The ***y*-intercept**
    - Find this by **substituting $x = 0$** to get $y = a_0$
  - The **roots**
    - You can find these by **factorising** or solving $y = 0$
  - The **shape**
    - This is determined by the **degree** (n) and the sign of the **leading coefficient** (a_n)

How does the multiplicity of a real root affect the graph of the polynomial?

- The **multiplicity** of a root is the number of times it is **repeated** when the polynomial is factorised
  - If $x = k$ is a root with **multiplicity *m*** then $(x - k)^m$ is a **factor** of the polynomial
- The graph either **crosses** the x-axis or **touches** the x-axis at a **root $x = k$** where k is a real number
  - If $x = k$ has **multiplicity 1** then the graph **crosses** the x-axis at $(k, 0)$
  - If $x = k$ has **multiplicity 2** then the graph has a **turning point** at $(k, 0)$ so **touches** the x-axis
    - If $x = k$ has **odd multiplicity** $m \geq 3$ then the graph has a **stationary point of inflection** at $(k, 0)$ so **crosses** the x-axis
    - If $x = k$ has **even multiplicity** $m \geq 4$ then the graph has a **turning point** at $(k, 0)$ so **touches** the x-axis

How do I determine the shape of the graph of the polynomial?

- Consider what happens as ***x* tends to ± ∞**
  - If a_n is **positive** and n is **even** then the graph **approaches from the top left** and **tends to the top right**
    - $\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = +\infty$
  - If a_n is **negative** and n is **even** then the graph **approaches from the bottom left** and **tends to the bottom right**
    - $\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = +\infty$
  - If a_n is **positive** and n is **odd** then the graph **approaches from the bottom left** and **tends to the top right**
    - $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to +\infty} f(x) = +\infty$
  - If a_n is **negative** and n is **odd** then the graph **approaches from the top left** and **tends to the bottom right**
    - $\lim_{x \to -\infty} f(x) = +\infty$ and $\lim_{x \to +\infty} f(x) = -\infty$
- Once you know the **shape**, the **real roots** and the ***y*-intercept** then you simply connect the points using a **smooth curve**
- There will be **at least one turning point** in-between each pair of roots
  - If the degree is n then there is **at most *n* – 1 stationary points (**some will be **turning points**)
    - Every real polynomial of **even degree** has **at least one turning point**
    - Every real polynomial of **odd degree bigger than 1** has **at least one point of inflection**
  - If it is a calculator paper then you can use your GDC to find the coordinates of the turning points
  - You won't need to find their location without a GDC unless the question asks you to

Worked example

a) The function f is defined by $f(x) = (x+1)(2x-1)(x-2)^2$. Sketch the graph of $y = f(x)$.

Find the y-intercept

$x = 0: \ y = (1)(-1)(-2)^2 = -4$

Find the roots and determine if graphs crosses or touches the x-axis

$(x+1)(2x-1)(x-2)^2$

$(-1,0)$ $(\frac{1}{2},0)$ $(2,0)$

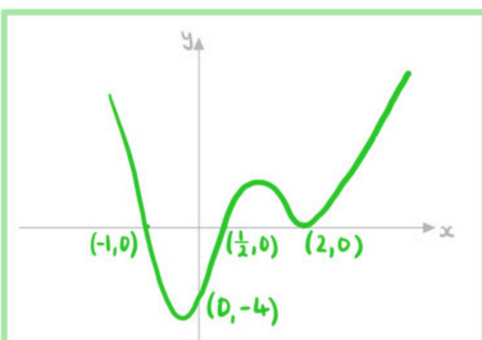
cross cross touch

Determine the shape by looking at the leading term

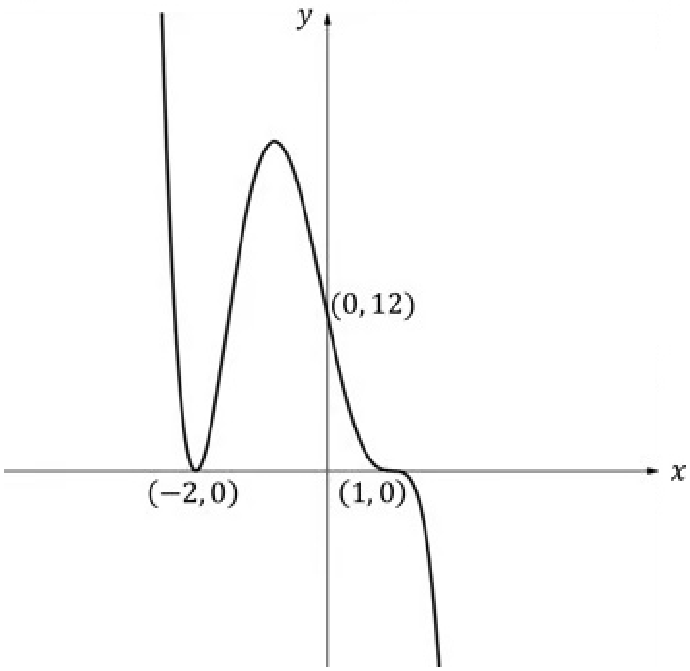
Leading term is $(x)(2x)(x)^2 = 2x^4$

As $x \to -\infty$ $y \to +\infty$

As $x \to +\infty$ $y \to +\infty$



b) The graph below shows a polynomial function. Find a possible equation of the polynomial.



Touches at $(-2,0)$ $\quad (x+2)^2$ is a factor

Point of inflection at $(1,0)$ $\quad (x-1)^3$ is a factor

Write in the form of : $\quad y = a(x+2)^2(x-1)^3$

Use the y-intercept to find a

$12 = a(2)^2(-1)^3 \quad \Rightarrow -4a = 12 \quad \therefore a = -3$

$$\boxed{y = -3(x+2)^2(x-1)^3}$$

Solving Polynomial Equations

What is "The Fundamental Theorem of Algebra"?

- Every **real polynomial** with degree n can be factorised into $\boldsymbol{n}$ **complex linear factors**
  - Some of which may be **repeated**
  - This means the polynomial will have n zeros (some may be repeats)
- Every **real polynomial** can be expressed as a product of **real linear factors** and **real irreducible quadratic factors**
  - An irreducible quadratic is where it **does not have real roots**
    - The **discriminant** will be negative: $\boldsymbol{b^2 - 4ac < 0}$
- If $a + bi$ ($b \neq 0$) is a **zero** of a **real polynomial** then its **complex conjugate** $a - bi$ is also a **zero**
- Every **real polynomial** of **odd degree** will have **at least one real zero**

How do I solve polynomial equations?

- Suppose you have an equation $P(x) = 0$ where $P(x)$ is a **real polynomial of degree** $\boldsymbol{n}$
  - $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ \dots + a_1 x + a_0$
- You may be given one zero or you might have to find a zero $x = k$ by substituting values into $P(x)$ until it equals 0
- If you know a **root** then you know a **factor**
  - If you know $\boldsymbol{x = k}$ **is a root** then $\boldsymbol{(x - k)}$ **is a factor**
  - If you know x **= a + bi is a root** then you know a **quadratic factor** $\boldsymbol{(x - (a + bi))(x - (a - bi))}$
    - Which can be written as $((x - a) - bi)((x - a) + bi)$ and **expanded quickly using difference of two squares**
- You can then **divide** $P(x)$ by this factor to get **another factor**
  - For example: dividing a cubic by a linear factor will give you a quadratic factor
- You then may be able to factorise this new factor

Worked example

Given that $x = \frac{1}{2}$ is a zero of the polynomial defined by $f(x) = 2x^3 - 3x^2 + 5x - 2$, find all three zeros of f.

$x = \frac{1}{2}$ is a root $\therefore$ $(2x-1)$ is a factor

Find the quadratic factor $(2x^3 - 3x^2 + 5x - 2) = (2x-1)(ax^2 + bx + c)$

Compare coefficients : $2x^3 = 2ax^3$ $\therefore a = 1$

$-2 = -c$ $\therefore c = 2$

$5x = 2cx - bx \Rightarrow 5 = 4 - b$ $\therefore b = -1$

Solve the quadratic : $x^2 - x + 2 = 0$

Formula booklet

Solutions of a quadratic equation	$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{-7}}{2} = \frac{1}{2} \pm \frac{\sqrt{7}}{2} i$$

$$\boxed{\text{Roots : } \frac{1}{2}, \frac{1}{2} + \frac{\sqrt{7}}{2} i, \frac{1}{2} - \frac{\sqrt{7}}{2} i}$$

2.7.4 Roots of Polynomials

Sum & Product of Roots

How do I find the sum & product of roots of polynomials?

- Suppose $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ ... + a_1 x + a_0$ is a **polynomial** of **degree** *n* with *n* roots $\alpha_1, \alpha_2, ..., \alpha_n$
  - The polynomial is written as $\sum_{r=0}^{n} a_r x^r = 0, \ a_n \neq 0$ in the **formula booklet**
  - a_n is the coefficient of the **leading term**
  - a_{n-1} is the coefficient of the $\mathbf{x^{n-1}}$ **term**
    - Be careful: this could be equal to zero
  - a_0 is the **constant term**
    - Be careful: this could be equal to zero
- In factorised form: $P(x) = a_n(x - \alpha_1)(x - \alpha_2)...(x - \alpha_n)$
  - Comparing coefficients of the $\mathbf{x^{n-1}}$ **term** and the **constant term** gives
    - $a_{n-1} = a_n(-\alpha_1 - \alpha_2 - ... - \alpha_n)$
    - $a_0 = a_n(-\alpha_1) \times (-\alpha_2) \times ... \times (-\alpha_n)$
- The **sum** of the roots is given by:
  - $\alpha_1 + \alpha_2 + \ ... + \alpha_n = -\frac{a_{n-1}}{a_n}$
- The **product** of the roots is given by:
  - $\alpha_1 \times \ \alpha_2 \times \ ... \times \alpha_n = \frac{(-1)^n a_0}{a_n}$
    - both of these formulae are in your **formula booklet**

How can I find unknowns if I am given the sum and/or product of the roots of a polynomial?

- If you know a complex root of a real polynomial then its **complex conjugate** is **another root**
- Form **two equations** using the roots
  - One using the **sum of the roots formula**
  - One using the **product of the roots formula**
- **Solve** for any unknowns

EXAM PAPERS PRACTICE

Worked example

$2-3\mathrm{i}$, $\frac{5}{3}\mathrm{i}$ and α are three roots of the equation $18x^5 - 9x^4 + 32x^3 + 794x^2 - 50x + k = 0$

.

a) Use the sum of all the roots to find the value of α.

It is a real polynomial so if $a+bi$ is a root then $a-bi$ is also a root

Roots: $2-3i,\ 2+3i,\ \frac{5}{3}i,\ -\frac{5}{3}i,\ \alpha$

Formula booklet

Sum & product of the roots of polynomial equations of the form $\sum_{r=0}^{n} a_r x^r = 0$	Sum is $\frac{-a_{n-1}}{a_n}$

$18x^5 - 9x^4 + 32x^3 + 794x^2 - 50x + k$

$a_n = 18 \quad a_{n-1} = -9$

$(2-3i)+(2+3i)+\left(\frac{5}{3}i\right)+\left(-\frac{5}{3}i\right)+\alpha = \frac{-(-9)}{18}$

$4+\alpha = \frac{1}{2}$

$\boxed{\alpha = -\frac{7}{2}}$

b) Use the product of all the roots to find the value of k.

Formula booklet

Sum & product of the roots of polynomial equations of the form $\sum_{r=0}^{n} a_r x^r = 0$	product is $\frac{(-1)^n a_0}{a_n}$

$18x^5 - 9x^4 + 32x^3 + 794x^2 - 50x + k$

$a_n = 18 \quad n = 5 \quad a_0 = k$

$(2-3i)(2+3i)\left(\frac{5}{3}i\right)\left(-\frac{5}{3}i\right)\left(-\frac{7}{2}\right) = \frac{(-1)^5 k}{18}$

$(13)\left(\frac{25}{9}\right)\left(-\frac{7}{2}\right) = \frac{-k}{18}$

$-\frac{2275}{18} = -\frac{k}{18}$

$\boxed{k = 2275}$