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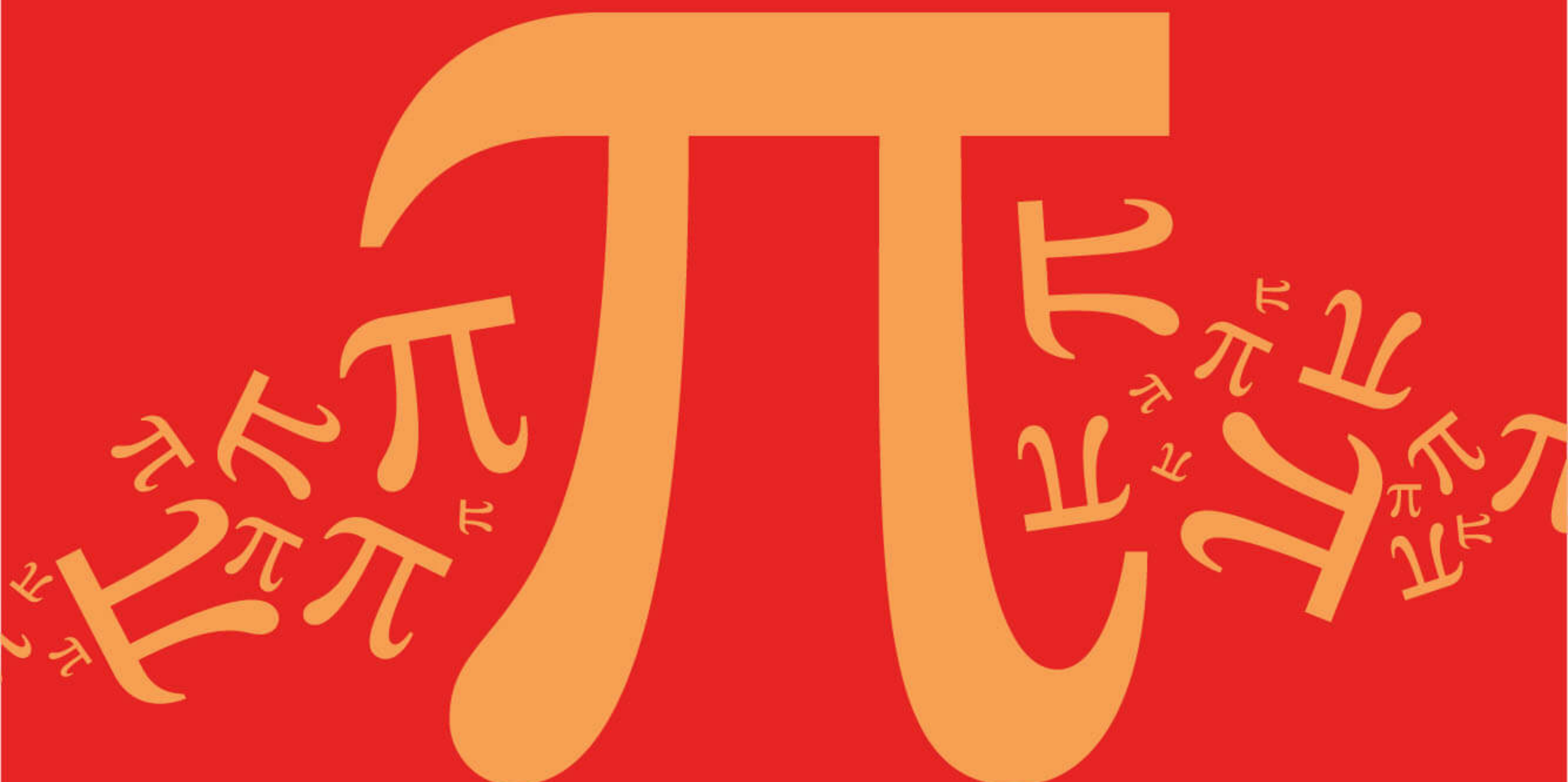
# 2.6 Further Modelling with Functions



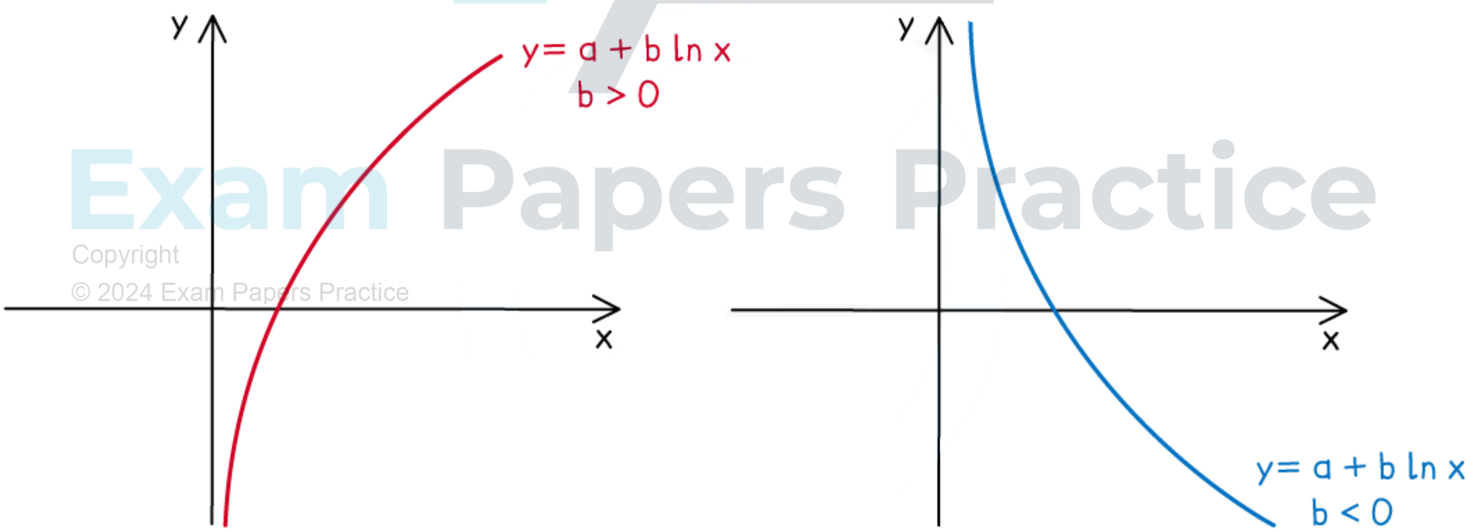
# IB Maths - Revision Notes

# 2.6.1 Properties of Further Graphs

## Logarithmic Functions & Graphs

### What are the key features of logarithmic graphs?

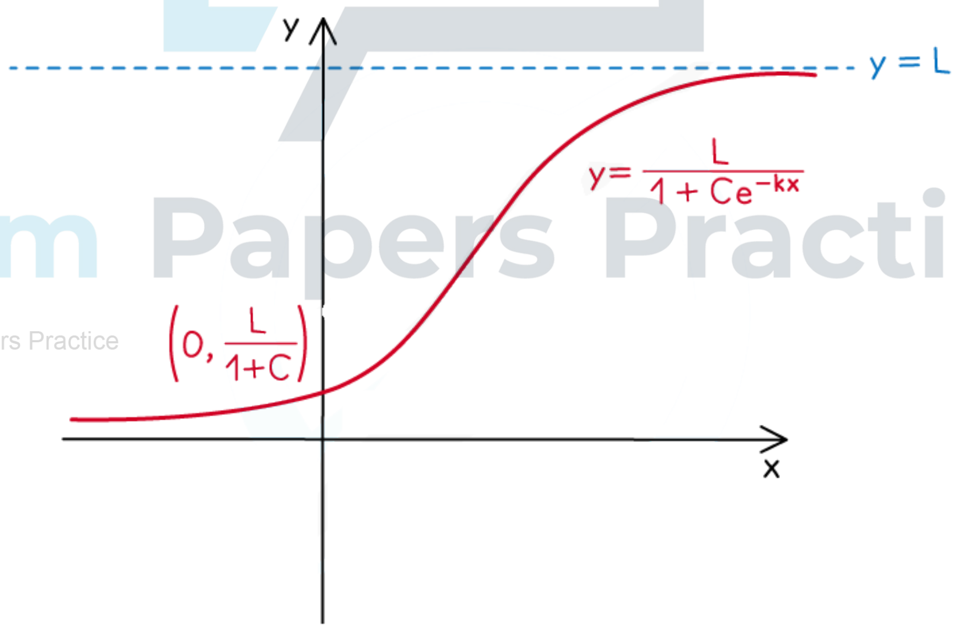
- A **logarithmic function** is of the form $f(x) = a + b\ln x,\ x > 0$
- Remember the natural logarithmic function $\ln x \equiv \log_e(x)$
  - This is the inverse of $f(x) = e^x$
    - $\ln(e^x) = x$ and $e^{\ln x} = x$
  - The graphs **will always** pass through the point (1, *a*)
  - The graphs **do not have a *y*-intercept**
    - The graphs have a **vertical asymptote** at the *y*-axis:
  - The graphs have **one root** at $\left(e^{-\frac{a}{b}}, 0\right)$
    - This can be found using your GDC
  - The graphs **do not have any minimum or maximum points**
  - The value of *b* determines whether the graph is increasing or decreasing
    - If *b* is positive then the graph is increasing
    - If *b* is negative then the graph is decreasing



## Logistic Functions & Graphs

### What are the key features of logistic graphs?

- A **logistic function** is of the form $f(x) = \frac{L}{1 + Ce^{-kx}}$
  - *L*, *C* & *k* are positive constants
- Its **domain** is the set of **all real values**
- Its **range** is the set of **real positive values less than *L***
- The *y*-intercept is at the point $\left(0, \frac{L}{1 + C}\right)$
- There are **no roots**
- There is a **horizontal asymptote** at $y = L$
  - This is called the carrying capacity
    - This is the upper limit of the function
    - For example: it could represent the limit of a population size
- There is a **horizontal asymptote** at $y = 0$
- The graph is **always increasing**



# 2.6.2 Natural Logarithmic Models

## Natural Logarithmic Models

### What are the parameters of natural logarithmic models?

- A **natural logarithmic model** is of the form $f(x) = a + b \ln x$
- The $a$ represents the value of the function when $x = 1$
- The $b$ determines the rate of change of the function
  - A bigger absolute value of $b$ leads to a faster rate of change

### What can be modelled as a natural logarithmic model?

- A **natural logarithmic model** can be used when the variable increases rapidly for a period followed by a much slower rate of increase with no limiting value
  - $M(I)$ is the magnitude of an earthquake with an intensity of $I$
  - $d(I)$ is the decibels measured of a noise with an intensity of $I$

### What are possible limitations a natural logarithmic model?

- A **natural logarithmic graph** is unbounded
  - However in real-life the variable might have a limiting value

#### Worked example

The sound intensity level, $L$, in decibels (dB) can be modelled by the function

$$L(I) = a + 8\ln I,$$

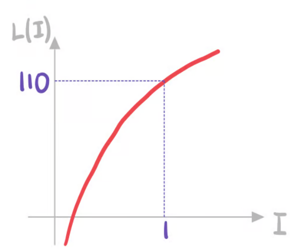
where $I$ is the sound intensity, in watts per square metre ($\text{Wm}^{-2}$).

a) Given that a sound intensity of $1\ \text{Wm}^{-2}$ produces a sound intensity level of 110 dB, write down the value of $a$.

Substitute $I = 1$ and $L = 110$

$110 = a + 8\ln 1$

$a = 110$



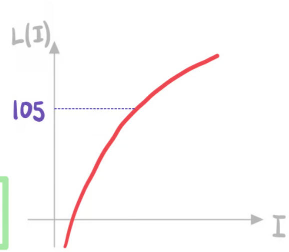

b) Find the sound intensity, in $\text{Wm}^{-2}$, of a car alarm that has a sound intensity level of 105 dB.

Use GDC to solve $L(I) = 105$

$110 + 8\ln I = 105$

$I = 0.535261...$

$I = 0.535\ \text{Wm}^{-2}$ (3sf)



# 2.6.3 Logistic Models

## Logistic Models

### What are the parameters of logistic models?

- A **logistic model** is of the form $f(x) = \dfrac{L}{1 + C\mathrm{e}^{-kx}}$
- The $L$ represents the limiting capacity
  - This is the value that the model tends to as $x$ gets large
- The $C$ (along with the $L$) helps to determine the initial value of the model
  - The initial value is given by $\dfrac{L}{1 + C}$
  - Once $L$ has been determined you can then determine $C$
- The $k$ determines the rate of increase of the model

### What can be modelled using a logistic model?

- A **logistic model** can be used when the variable initially increases exponentially and then tends towards a limit
  - $H(t)$ is the height of a giraffe $t$ weeks after birth
  - $P(t)$ is the number of bacteria on an apple $t$ seconds after removing from protective packaging
  - $P(t)$ is the population of rabbits in a woodlands area $t$ weeks after releasing an initial amount into the area



### What are possible limitations of a logistic model?

- A logistic graph is **bounded** by the limit $L$
  - However in real-life the variable might be unbounded
    - For example: the cumulative total number of births in a town over time
- A logistic graph is **always increasing**
  - However in real-life there could be periods where the variable decreased or fluctuates

## Worked example

The number of fish in a lake, $F$, can be modelled by the function

$$F(t) = \frac{800}{1 + Ce^{-0.6t}}$$

where $t$ is the number of months after fish were introduced to the lake.

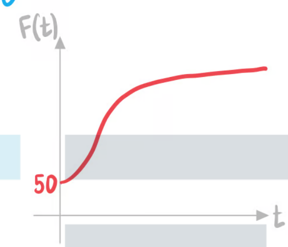
a) Initially, 50 fish were introduced to the lake. Find the value of $C$.

Substitute $t = 0$ and $F = 50$

$$50 = \frac{800}{1 + Ce^0}$$

$$50 = \frac{800}{1 + C}$$

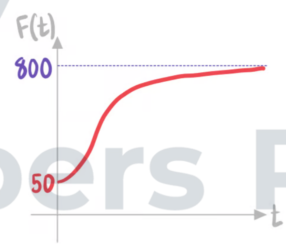
**$C = 15$**




b) Write down the limiting capacity for the number of fish in the lake.

Find the horizontal asymptote

**Limiting capacity is 800**





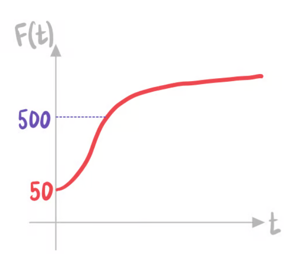

c) Calculate the number of months it takes until there are 500 fish in the lake.

Solve $F(t) = 500$

$$\frac{800}{1 + 15e^{-0.6t}} = 500$$

$$t = 5.3647...$$

**5.36 months**



# 2.6.4 Piecewise Models

## Linear Piecewise Models

### What are the parameters of a piecewise linear model?

- A **piecewise linear model** is made up of multiple linear models $f_i(x) = m_i x + c_i$
- For each linear model there will be
  - The rate of change for that interval $m_i$
  - The value if the independent variable was not present $c_i$

### What can be modelled as a piecewise linear model?

- Piecewise linear models can be used when the rate of change of a function changes for different intervals
  - These commonly apply when there are different tariffs or levels of charges
- Anything with a constant rate of change for set intervals
  - $C(d)$ is the taxi charge for a journey of $d$ km
    - The charge might double after midnight
  - $R(d)$ is the rental fee for a car used for $d$ days
    - The daily fee might triple if the car is rented over bank holidays
  - $s(t)$ is the speed of a car travelling for $t$ seconds with constant acceleration
    - The car might reach a maximum speed

### What are possible limitations of a piecewise linear model?



- Piecewise linear models have a constant rate of change (represented by a straight line) in each interval
  - In real-life this might not be the case
  - The data in some intervals might have a continuously variable rate of change (represented by a curve) rather than a constant rate
  - Or the transition from one constant rate of change to another may be gradual - i.e. a curve rather than a sudden change in gradient

> **Exam Tip**
>
> - Make sure that you know how to plot a piecewise model on your GDC

## Worked example

The total monthly charge, £ $C$, of phone bill can be modelled by the function

$$C(m) = \begin{cases} 10 + 0.02m & 0 \le m \le 100 \\ 9 + 0.03m & m > 100 \end{cases},$$

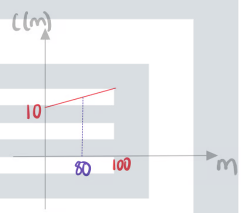
where $m$ is the number of minutes used.

a) Find the total monthly charge if 80 minutes have been used.

Substitute $m = 80$ into the first function

$C(80) = 10 + 0.02(80)$

**£11.60**




b) Given that the total monthly charge is £16.59, find the number of minutes that were used.

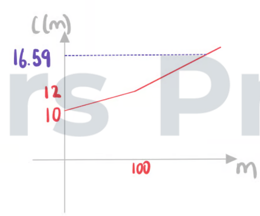
Substitute $C = 16.59$ into the second function

$16.59 = 9 + 0.03m$

$0.03m = 7.59$

$m = \frac{7.59}{0.03}$

**253 minutes**



## Non-Linear Piecewise Models

### What are the parameters of non-linear piecewise models?

- A **non-linear piecewise model** is made up of multiple functions $f_i(x)$
  - Each function will be defined for a range of values of *x*
- The individual functions can contain **any function**
  - For example: quadratic, cubic, exponential, etc
- When graphed the individual functions should join to make a continuous graph
  - This fact can be used to find unknown parameters
    - If $f(x) = \begin{cases} f_1(x) & a \le x < b \\ f_2(x) & b \le x < c \end{cases}$ then $f_1(b) = f_2(b)$

### What can be modelled as a non-linear piecewise model?

- Piecewise models can be used when different functions are needed to represent the output for different intervals of the variable
  - *S*(*x*) is the standardised score on a test with *x* raw marks
    - For small values of *x* there might be a quadratic model
    - For large values of *x* there might be a linear model
  - *H*(*t*) is the height of water in a bathtub with after *t* minutes
    - Initially a cubic model might be a appropriate if the bottom of the bathtub is curved
    - Then a linear model might be a appropriate if the sides of top of the bathtub has the shape of a prism

### What are possible limitations a non-linear piecewise model?

- Piecewise models can be used to model real-life accurately
- Piecewise models can be difficult to analyse or apply mathematical techniques to



### Exam Tip

- Read and re-read the question carefully, try to get involved in the context of the question!
- Pay particular attention to the domain of each section, if it is not given think carefully about any restrictions there may be as a result of the context of the question
- If sketching a piecewise function, make sure to include the coordinates of all key points including the point at which two sections of the piecewise model meet

## Worked example

Jamie is running a race. His distance from the start, $x$ metres, can be modelled by the function

$$x(t) = \begin{cases} 3t & 0 \le t < 5 \\ 125 - a(t-15)^2 & 5 \le t < 15 \end{cases}$$

where $t$ is the time, in seconds, elapsed since the start of the race.

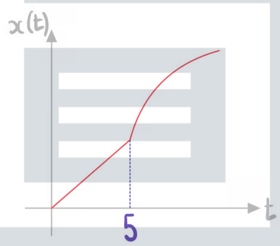
a) Find the value of $a$.

The function is continuous at $t = 5$

$3(5) = 125 - a(5-15)^2$

$15 = 125 - 100a$

**$a = 1.1$**




b) Find the time taken for Jamie to reach 100 metres from the start.

Decide which function to use

$x(5) = 15$

$100 > 15$

Solve $x(t) = 100$

$125 - 1.1(t-15)^2 = 100$

$t = 10.23...$ or $t = 19.76...$

Reject as $5 \le t < 15$

**$t = 10.2$ seconds**