DP IB Maths: AA HL

2.5 Reciprocal & Rational Functions

Contents

2.5.1 Reciprocal & Rational Functions

Reciprocal Functions & Graphs

What is the reciprocal function?

- The **reciprocal function** is defined by $f(x)=\frac{1}{x},\ x\neq 0$
- Its **domain** is the set of **all real values except 0**
- Its **range** is the set of **all real values except 0**
- The reciprocal function has a **self-inverse** nature
  - $f^{-1}(x)=f(x)$
  - $(f\circ f)(x)=x$

What are the key features of the reciprocal graph?

- The graph **does not have a *y*-intercept**
- The graph **does not have any roots**
- The graph has **two asymptotes**
  - A **horizontal** asymptote at the *x*-axis: $y=0$
    - This is the **limiting value** when the absolute value of *x* gets very large
  - A **vertical** asymptote at the *y*-axis: $x=0$
    - This is the value that causes the **denominator to be zero**
- The graph has **two axes of symmetry**
  - $y=x$
  - $y=-x$
- The graph **does not have any minimum or maximum points**

Linear Rational Functions & Graphs

What is a rational function with linear terms?

- A **(linear) rational function** is of the form $f(x) = \dfrac{ax+b}{cx+d},\ x \neq -\dfrac{d}{c}$
- Its **domain** is the set of **all real values except** $-\dfrac{d}{c}$
- Its **range** is the set of **all real values except** $\dfrac{a}{c}$
- The **reciprocal function** is a **special case** of a rational function

What are the key features of linear rational graphs?

- The graph has a ***y*-intercept** at $\left(0,\ \dfrac{b}{d}\right)$ provided $d \neq 0$
- The graph has **one root** at $\left(-\dfrac{b}{a},\ 0\right)$ provided $a \neq 0$
- The graph has **two asymptotes**
  - A **horizontal** asymptote: $y = \dfrac{a}{c}$
    - This is the **limiting value** when the absolute value of *x* gets very large
  - A **vertical** asymptote: $x = -\dfrac{d}{c}$
    - This is the value that causes the **denominator to be zero**
- The graph **does not have any minimum or maximum points**
- If you are asked to **sketch or draw** a rational graph:
  - Give the **coordinates** of any **intercepts** with the axes
  - Give the **equations** of the **asymptotes**

Worked example

The function f is defined by $f(x) = \dfrac{10-5x}{x+2}$ for $x \neq -2$.

a) Write down the equation of

(i) the vertical asymptote of the graph of f,

(ii) the horizontal asymptote of the graph of f.

(i) Vertical asymptote is when denominator equals zero

$x+2=0 \qquad \boxed{x=-2}$

(ii) Horizontal asymptote is limiting value as x gets large

$\lim\limits_{x\to\infty} \dfrac{10-5x}{x+2} = \lim\limits_{x\to\infty} \dfrac{-5\cancel{x}}{\cancel{x}} \qquad \boxed{y=-5}$

b) Find the coordinates of the intercepts of the graph of f with the axes.

y-intercept occurs when $x=0$

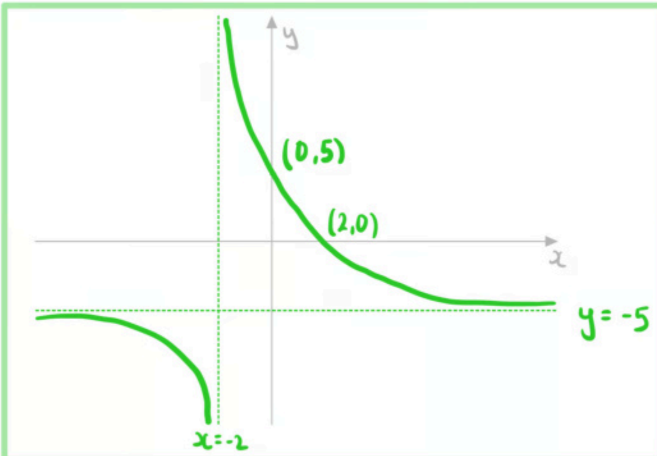
$y = \dfrac{10-5(0)}{0+2} = 5 \qquad \boxed{(0,5)}$

x-intercept occurs when $y=0$

$\dfrac{10-5x}{x+2} = 0 \Rightarrow 10-5x=0 \Rightarrow x=2 \qquad \boxed{(2,0)}$

c) Sketch the graph of f.

Include asymptotes and intercepts



Quadratic Rational Functions & Graphs

How do I sketch the graph of a rational function where the terms are not linear?

- A rational function can be written $f(x) = \dfrac{g(x)}{h(x)}$
  - Where *g* and *h* are polynomials
- To find the ***y*-intercept** evaluate $\dfrac{g(0)}{h(0)}$
- To find the ***x*-intercept(s)** solve $g(x) = 0$
- To find the equations of the **vertical asymptote(s)** solve $h(x) = 0$
- There will also be an **asymptote** determined by what *f*(*x*) tends to as *x* approaches infinity
  - In this course it will be either:
    - **Horizontal**
    - **Oblique (a slanted line)**
  - This can be found by writing $g(x)$ in the form $h(x)Q(x) + r(x)$
    - You can do this by **polynomial division** or **comparing coefficients**
  - The function then tends to the curve $y = Q(x)$

What are the key features of rational graphs: quadratic over linear?

- For the rational function of the form $f(x) = \dfrac{ax^2 + bx + c}{dx + e}$
- The graph has a ***y*-intercept** at $\left(0, \dfrac{c}{e}\right)$ provided $e \neq 0$
- The graph can have **0, 1 or 2 roots**
  - They are the solutions to $ax^2 + bx + c = 0$
- The graph has **one vertical asymptote** $x = -\dfrac{e}{d}$
- The graph has an **oblique asymptote** $y = px + q$
  - Which can be found by writing $ax^2 + bx + c$ in the form $(dx + e)(px + q) + r$
    - Where *p*, *q*, *r* are constants
    - This can be done by **polynomial division** or **comparing coefficients**

What are the key features of rational graphs: linear over quadratic?

- For the rational function of the form $f(x) = \dfrac{ax + b}{cx^2 + dx + e}$
- The graph has a ***y*-intercept** at $\left(0, \dfrac{b}{e}\right)$ provided $e \neq 0$
- The graph has **one root** at $x = -\dfrac{b}{a}$
- The graph has can have **0, 1 or 2 vertical asymptotes**
  - They are the solutions to $cx^2 + dx + e = 0$
- The graph has a **horizontal asymptote**

Worked example

The function f is defined by $f(x) = \dfrac{2x^2 + 5x - 3}{x + 1}$ for $x \neq -1$.

a) (i) Show that $\dfrac{2x^2 + 5x - 3}{x + 1} = px + q + \dfrac{r}{x + 1}$ for constants p, q and r which are to be found.

(ii) Hence write down the equation of the oblique asymptote of the graph of f.

(i) Write $2x^2 + 5x - 3$ as $(x+1)(px+q) + r$

$2x^2 + 5x - 3 = px^2 + qx + px + q + r$

Compare coefficients

x^2	x	constant
$2 = p$	$5 = q + p$	$-3 = q + r$
$\therefore p = 2$	$\therefore q = 3$	$\therefore r = -6$

$$\boxed{\frac{2x^2+5x-3}{x+1} = \frac{(x+1)(2x+3)-6}{x+1} = 2x + 3 - \frac{6}{x+1}}$$

(ii) $\boxed{y = 2x + 3}$

b) Find the coordinates of the intercepts of the graph of f with the axes.

y-intercept occurs when $x = 0$

$y = \dfrac{2(0)^2 + 5(0) - 3}{(0) + 1} = -3$ $\boxed{(0, -3)}$

x-intercept occurs when $y = 0$

$\dfrac{2x^2+5x-3}{x+1} = 0 \Rightarrow 2x^2 + 5x - 3 = 0 \Rightarrow (2x-1)(x+3) \Rightarrow x = 0.5$ or $x = -3$

$\boxed{(0.5, 0) \text{ and } (-3, 0)}$

c) Sketch the graph of f.

Vertical asymptote when denominator is zero $x = -1$

Include asymptotes and intercepts