EXAM PAPERS PRACTICE

2.5 Transformations of Graphs

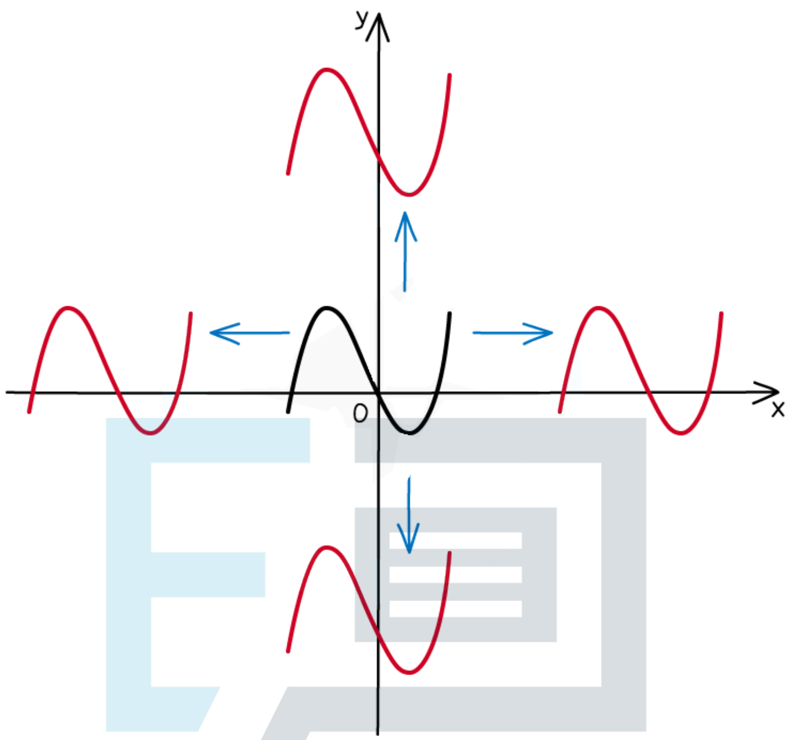
IB Maths - Revision Notes

2.5.1 Translations of Graphs

Translations of Graphs

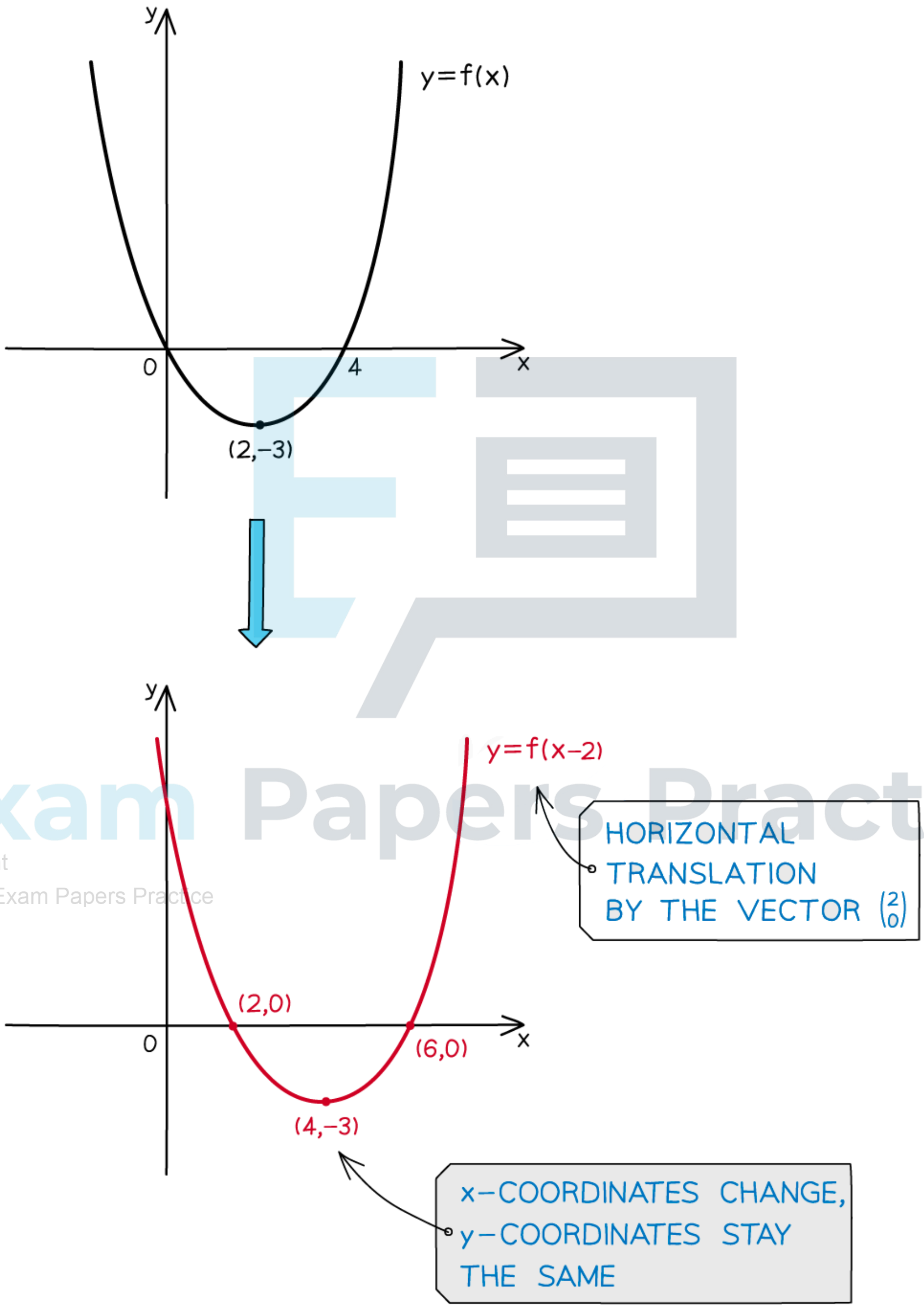
What are translations of graphs?

- When you alter a function in certain ways, the effects on the graph of the function can be described by **geometrical transformations**
- For a **translation**:
  - the graph is **moved** (up or down, left or right) in the xy plane
    - Its position **changes**
  - the shape, size, and orientation of the graph remain **unchanged**
- A particular translation (how far left/right, how far up/down) is specified by a **translation vector** $\begin{pmatrix} x \\ y \end{pmatrix}$:
  - x is the **horizontal** displacement
    - **Positive** moves **right**
    - **Negative** moves **left**
  - y is the **vertical** displacement
    - **Positive** moves **up**
    - **Negative** moves **down**



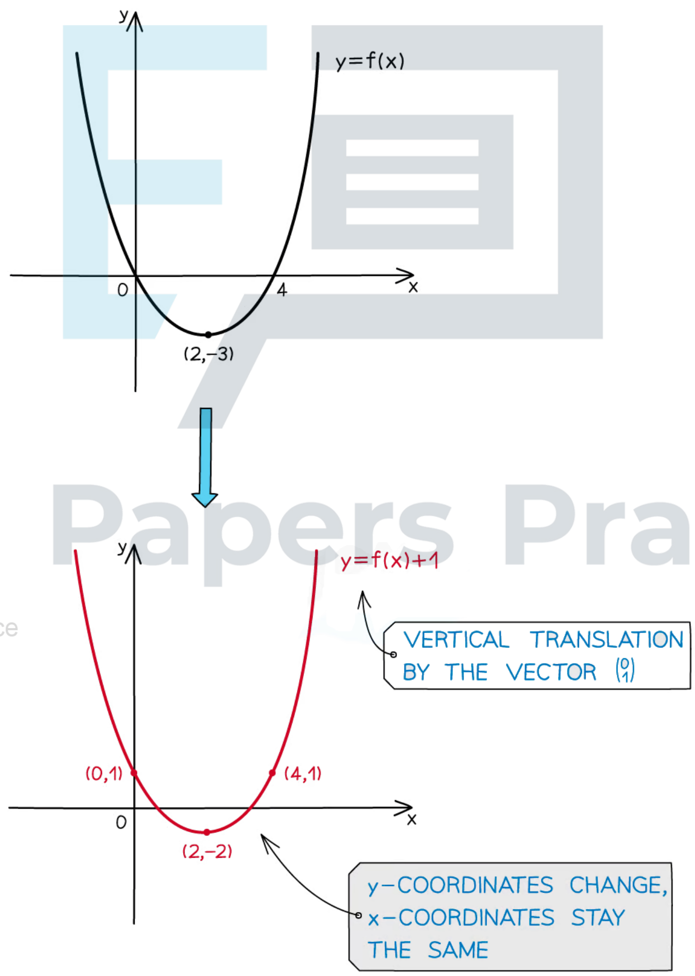
What effects do horizontal translations have on the graphs and functions?

- A **horizontal translation** of the graph $y = f(x)$ by the vector $\begin{pmatrix} a \\ 0 \end{pmatrix}$ is represented by
    - $y = f(x - a)$
- The ***x*-coordinates change**
    - The value *a* is **subtracted** from them
- The ***y*-coordinates stay the same**
- The coordinates (x, y) become $(x + a, y)$
- **Horizontal** asymptotes **stay the same**
- **Vertical** asymptotes **change**
    - $x = k$ becomes $x = k + a$


y
y=f(x)
0
4
x
(2,−3)
y
y=f(x−2)
HORIZONTAL TRANSLATION BY THE VECTOR $\binom{2}{0}$
(2,0)
0
(6,0)
x
(4,−3)
x−COORDINATES CHANGE, y−COORDINATES STAY THE SAME

What effects do vertical translations have on the graphs and functions?

- A **vertical translation** of the graph $y = f(x)$ by the vector $\begin{pmatrix} 0 \\ b \end{pmatrix}$ is represented by
  - $y - b = f(x)$
  - This is often rearranged to $y = f(x) + b$
- The ***x*-coordinates stay the same**
- The ***y*-coordinates change**
  - The value b is **added** to them
- The coordinates (x, y) become $(x, y + b)$
- **Horizontal** asymptotes **change**
  - $y = k$ becomes $y = k + b$
- **Vertical** asymptotes **stay the same**

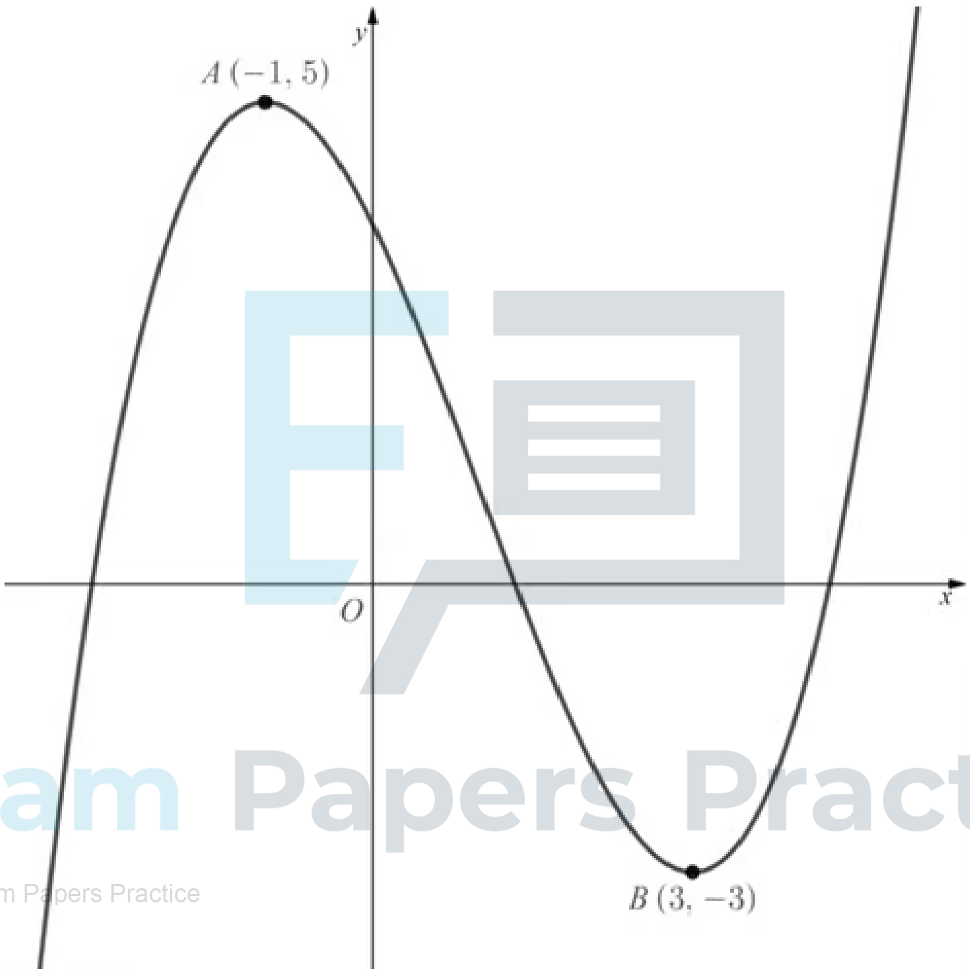


Exam Tip

- To get full marks in an exam make sure you use correct mathematical terminology
  - For example: Translate by the vector $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$

Worked example

The diagram below shows the graph of $y = f(x)$.



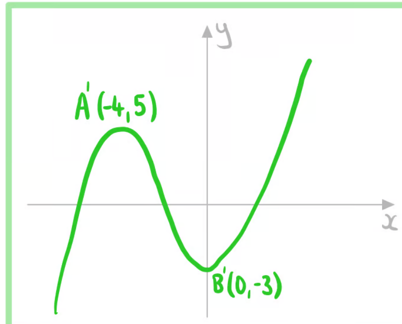
a) Sketch the graph of $y = f(x+3)$.

$y = f(x+k)$ translation by $\begin{pmatrix} -k \\ 0 \end{pmatrix}$

Translate $y = f(x)$ by $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

A becomes $(-4, 5)$

B becomes $(0, -3)$



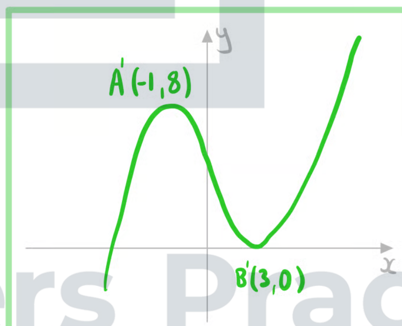
b) Sketch the graph of $y = f(x) + 3$.

$y = f(x)+k$ translation by $\begin{pmatrix} 0 \\ k \end{pmatrix}$

Translate $y = f(x)$ by $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

A becomes $(-1, 8)$

B becomes $(3, 0)$

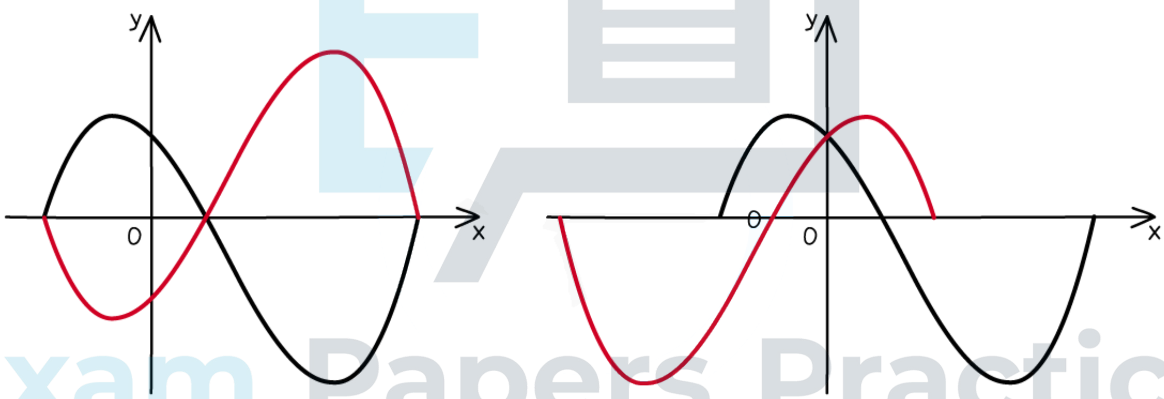


2.5.2 Reflections of Graphs

Reflections of Graphs

What are reflections of graphs?

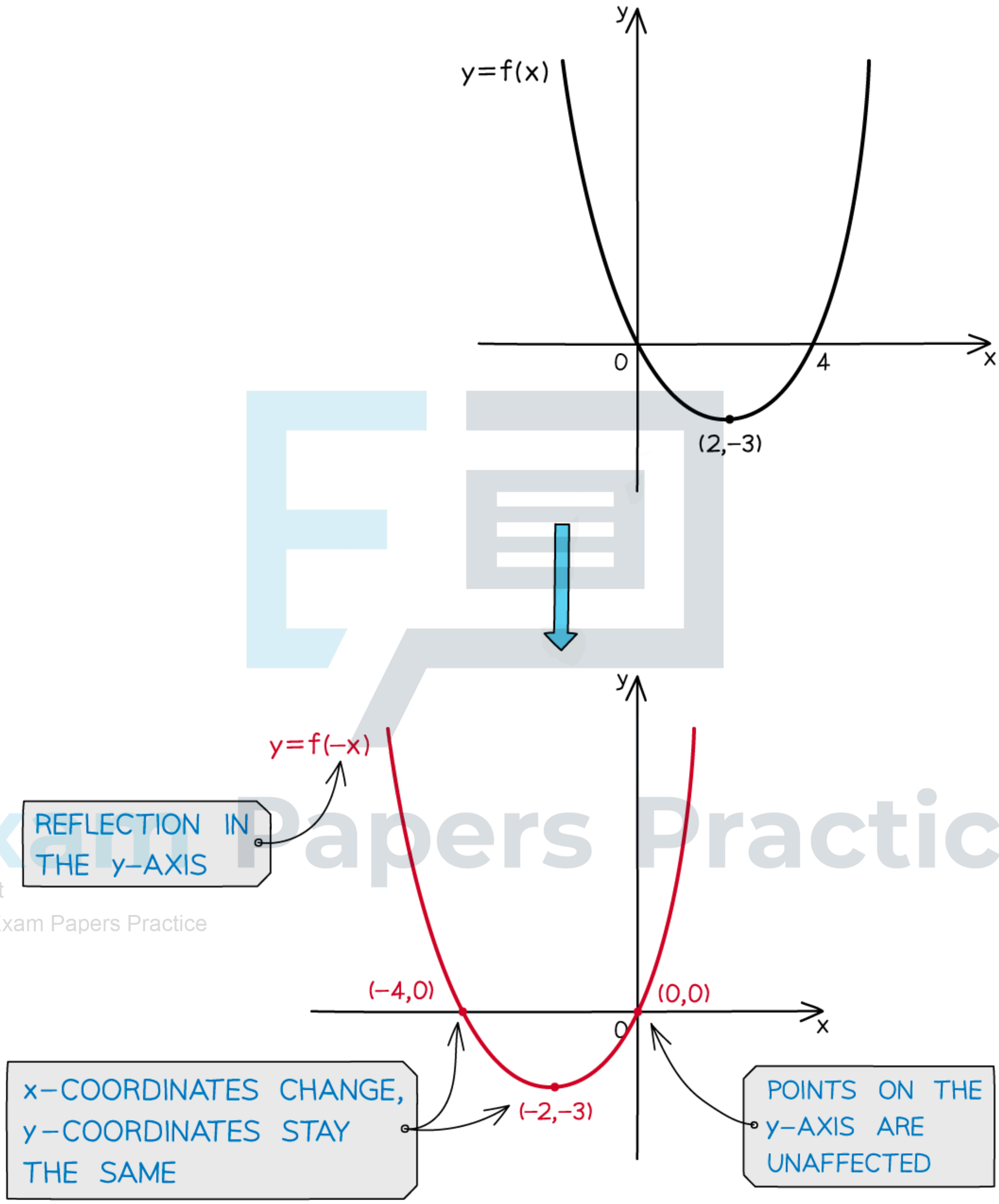
- When you alter a function in certain ways, the effects on the graph of the function can be described by **geometrical transformations**
- For a **reflection**:
  - the graph is **flipped** about one of the coordinate axes
    - Its orientation **changes**
  - the size of the graph remains **unchanged**
- A particular reflection is specified by an **axis of symmetry**:
  - $y = 0$
    - This is the *x*-axis
  - $x = 0$
    - This is the *y*-axis



What effects do horizontal reflections have on the graphs and functions?

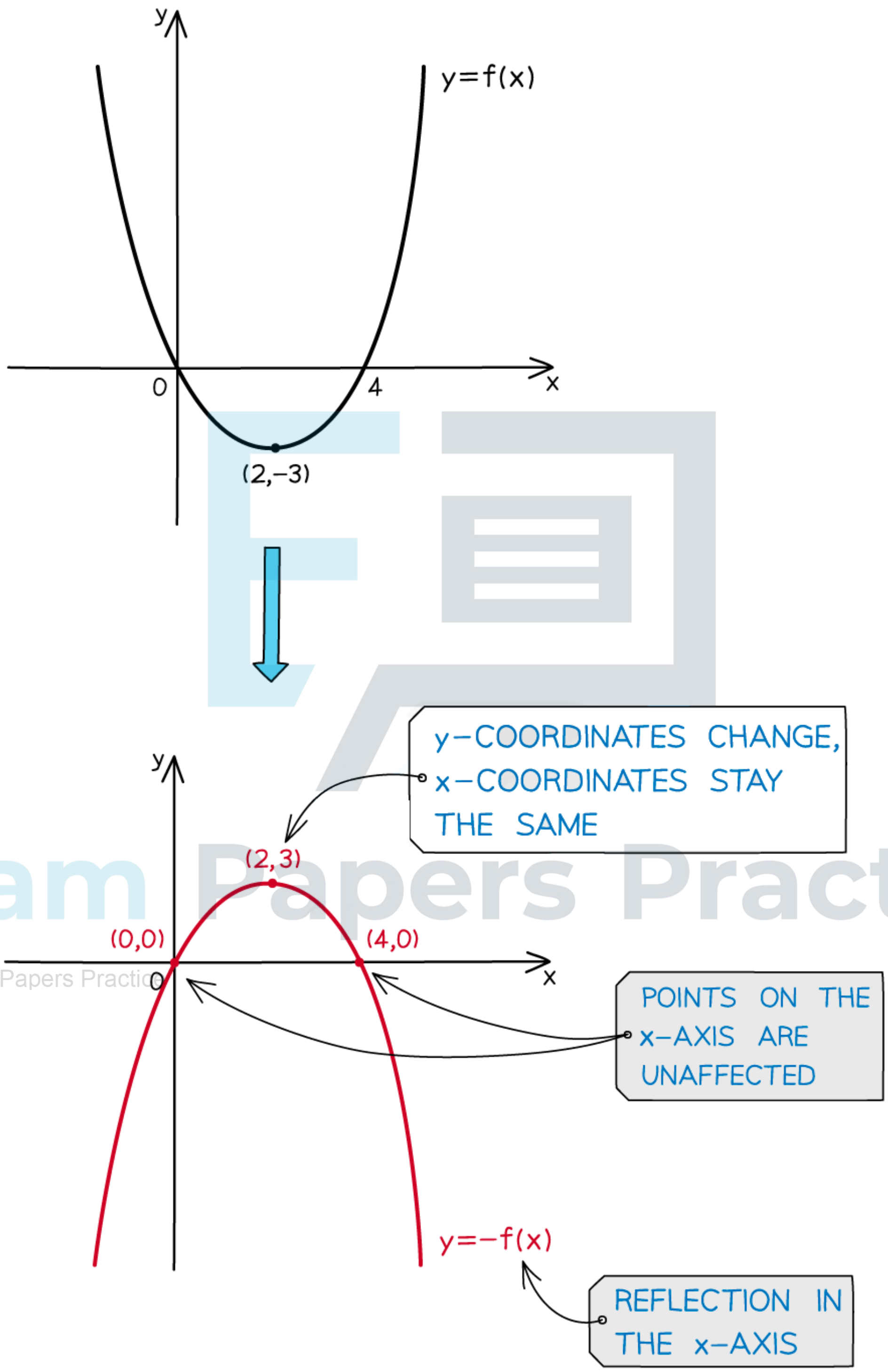
- A **horizontal reflection** of the graph $y = f(x)$ about the *y*-axis is represented by
  - $y = f(-x)$
- The ***x*-coordinates change**
  - Their **sign** changes
- The ***y*-coordinates stay the same**

- The coordinates (x, y) become $(-x, y)$
- **Horizontal** asymptotes **stay the same**
- **Vertical** asymptotes **change**
  - $x = k$ becomes $x = -k$


y=f(x)
y
x
0
4
(2,−3)
y=f(−x)
REFLECTION IN THE y−AXIS
y
x
(−4,0)
(0,0)
0
(−2,−3)
x−COORDINATES CHANGE, y−COORDINATES STAY THE SAME
POINTS ON THE y−AXIS ARE UNAFFECTED

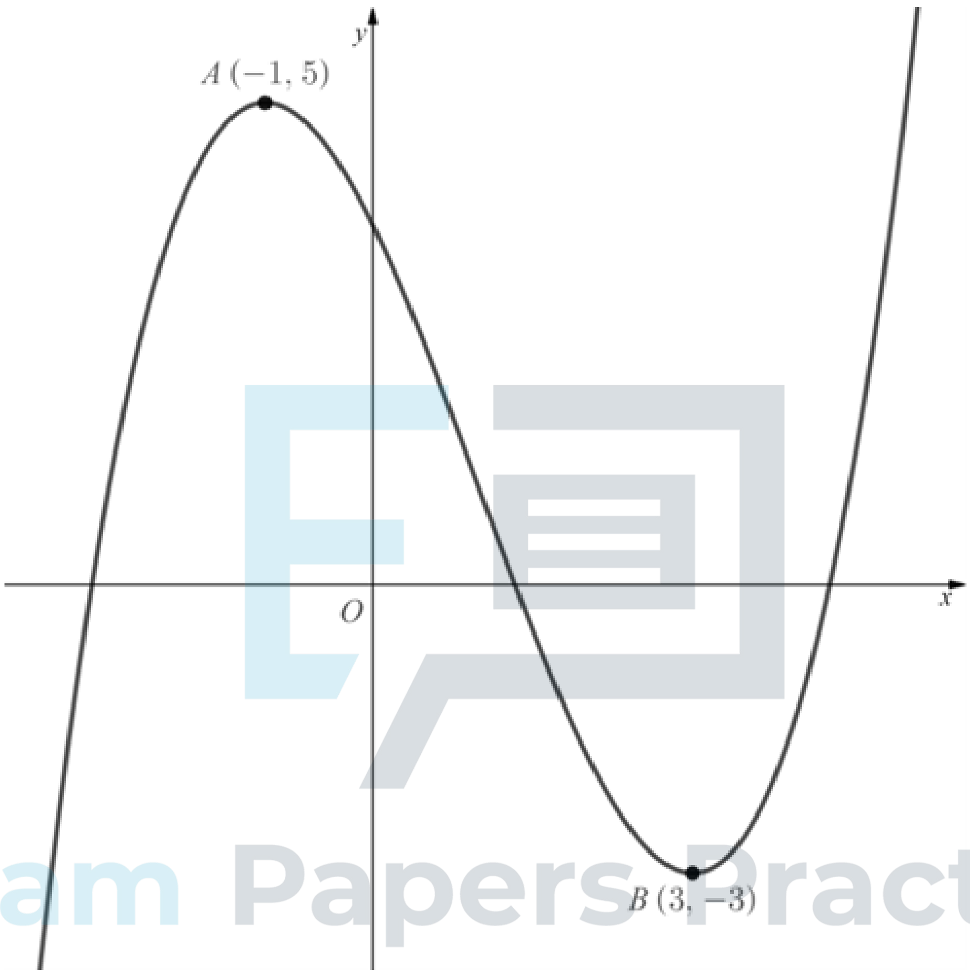
What effects do vertical reflections have on the graphs and functions?

- A **vertical reflection** of the graph $y = f(x)$ about the *x*-axis is represented by
  - $-y = f(x)$
  - This is often rearranged to $y = -f(x)$
- The ***x*-coordinates stay the same**
- The ***y*-coordinates change**
  - Their **sign** changes
- The coordinates (x, y) become $(x, -y)$
- **Horizontal** asymptotes **change**
  - $y = k$ becomes $y = -k$
- **Vertical** asymptotes **stay the same**


y
y=f(x)
0
4
x
(2,−3)
y
y−COORDINATES CHANGE, x−COORDINATES STAY THE SAME
(2,3)
(0,0)
(4,0)
0
x
POINTS ON THE x−AXIS ARE UNAFFECTED
y=−f(x)
REFLECTION IN THE x−AXIS

Worked example

The diagram below shows the graph of $y = f(x)$.

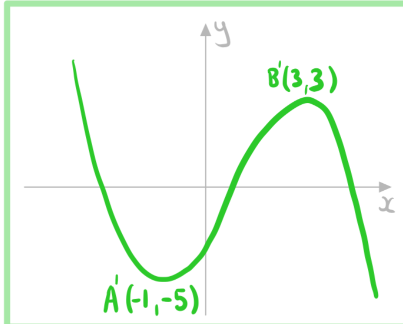


a) Sketch the graph of $y = -f(x)$.

$y = -f(x)$ reflection in x-axis

A becomes $(-1, -5)$

B becomes $(3, 3)$

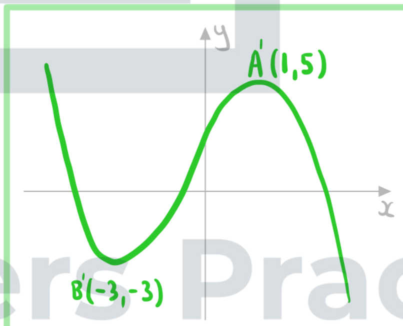


b) Sketch the graph of $y = f(-x)$.

$y = f(-x)$ reflection in y-axis

A becomes $(1, 5)$

B becomes $(-3, -3)$

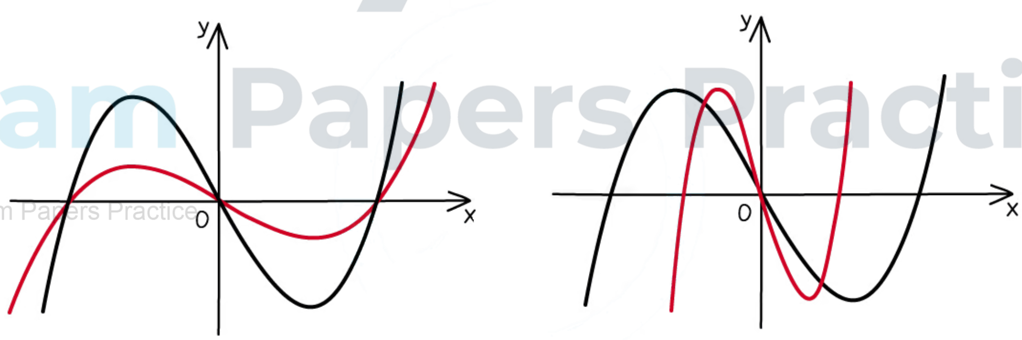


2.5.3 Stretches of Graphs

Stretches of Graphs

What are stretches of graphs?

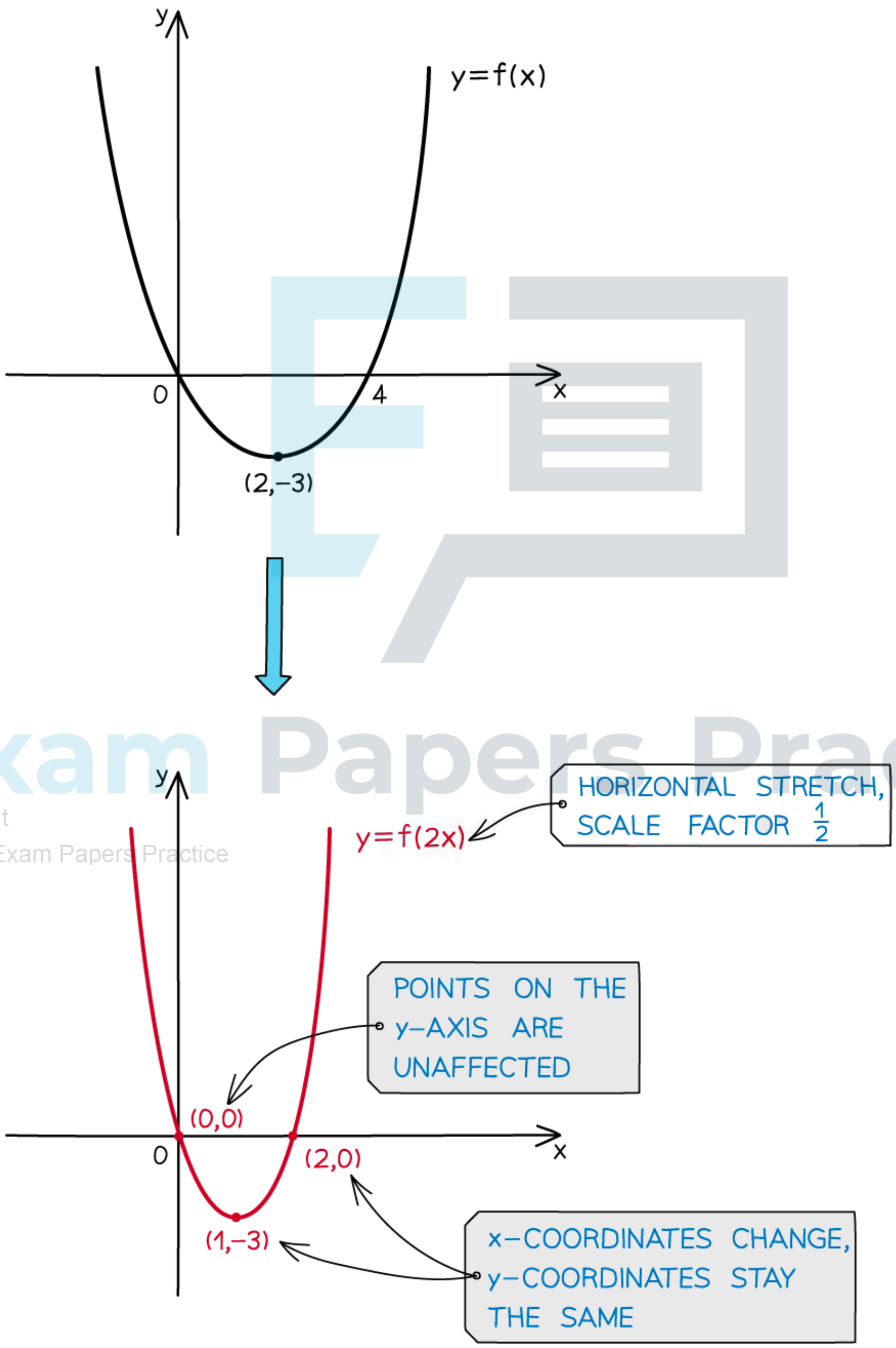
- When you alter a function in certain ways, the effects on the graph of the function can be described by **geometrical transformations**
- For a **stretch**:
  - the graph is **stretched** about one of the coordinate axes by a scale factor
    - Its size **changes**
  - the orientation of the graph remains **unchanged**
- A particular stretch is specified by a **coordinate axis** and a **scale factor**:
  - The **distance** between a **point** on the graph and the **specified coordinate axis** is **multiplied** by the **constant scale factor**
  - The graph is stretched in the **direction** which is **parallel** to the **other coordinate axis**
  - For scale factors **bigger than 1**
    - the points on the graph get **further away** from the **specified coordinate axis**
  - For scale factors **between 0 and 1**
    - the points on the graph get **closer** to the **specified coordinate axis**
    - This is also sometimes called a **compression** but in your exam you must use the term **stretch** with the appropriate scale factor



What effects do horizontal stretches have on the graphs and functions?

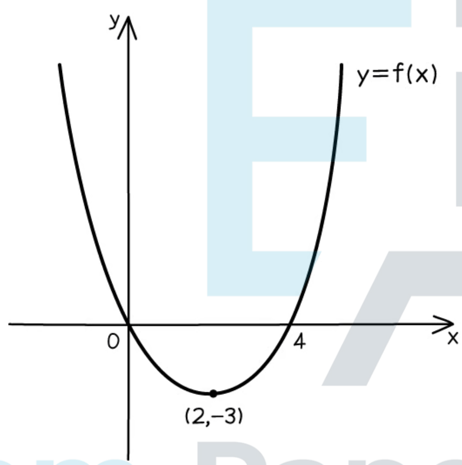
- A **horizontal stretch** of the graph $y = f(x)$ by a scale factor q centred about the y-axis is represented by
  - $y = f\left(\dfrac{x}{q}\right)$
- The **x-coordinates change**
  - They are **divided** by q
- The **y-coordinates stay the same**

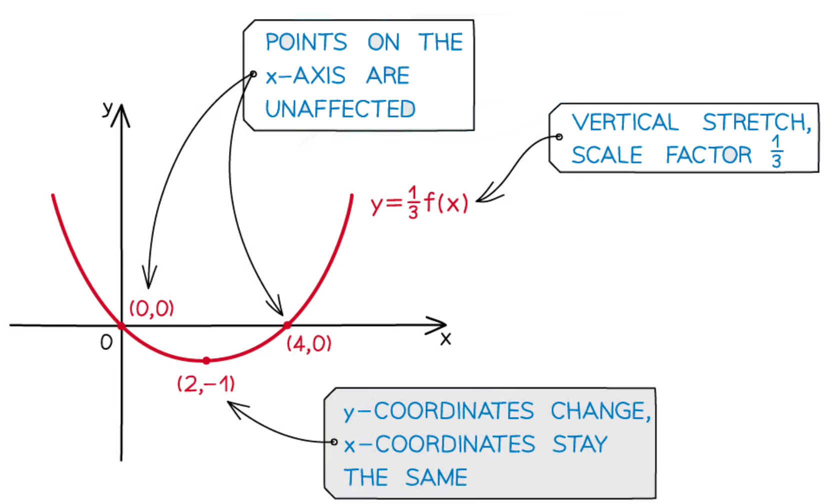
- The coordinates (x, y) become (qx, y)
- **Horizontal** asymptotes **stay the same**
- **Vertical** asymptotes **change**
  - $x = k$ becomes $x = qk$



What effects do vertical stretches have on the graphs and functions?

- A **vertical stretch** of the graph $y = f(x)$ by a scale factor *p* centred about the *x*-axis is represented by
  - $\frac{y}{p} = f(x)$
  - This is often rearranged to $y = pf(x)$
- The ***x*-coordinates stay the same**
- The ***y*-coordinates change**
  - They are **multiplied** by *p*
- The coordinates (x, y) become (x, py)
- **Horizontal** asymptotes **change**
  - $y = k$ becomes $y = pk$
- **Vertical** asymptotes **stay the same**



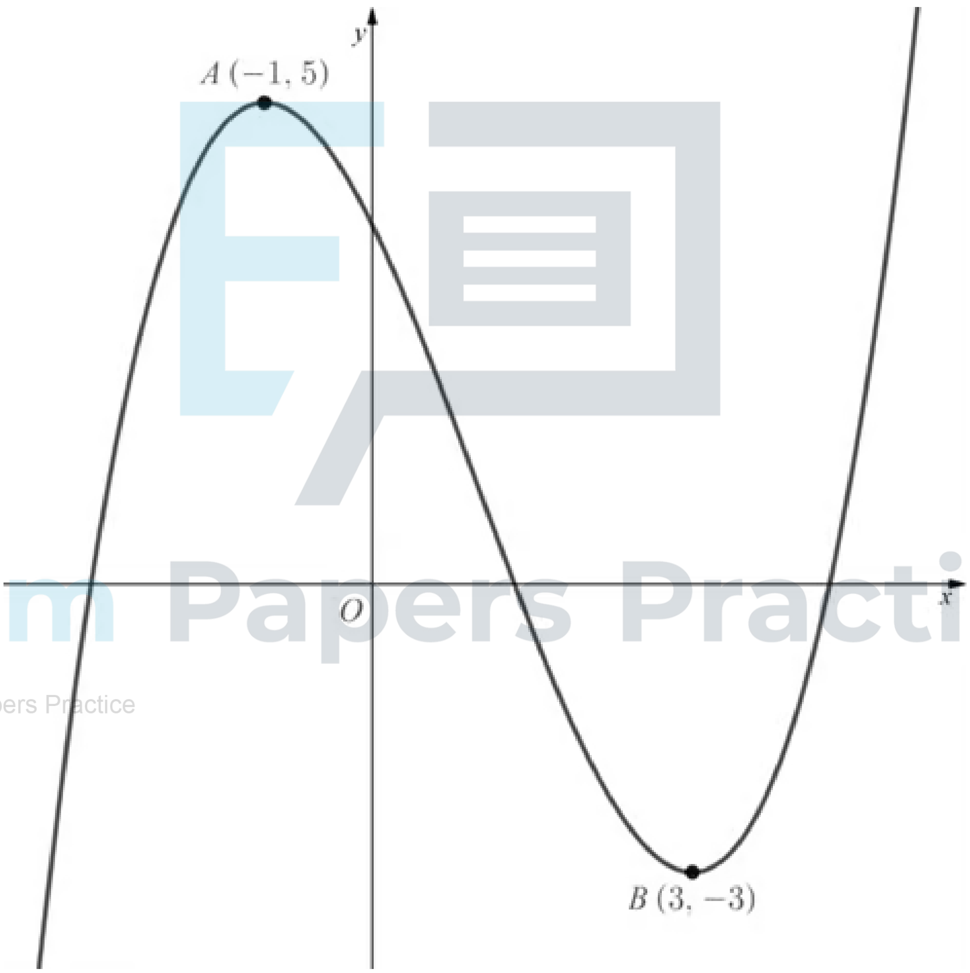


Exam Tip

- To get full marks in an exam make sure you use correct mathematical terminology
  - For example: Stretch vertically by scale factor ½
  - Do not use the word "compress" in your exam

Worked example

The diagram below shows the graph of $y = f(x)$.



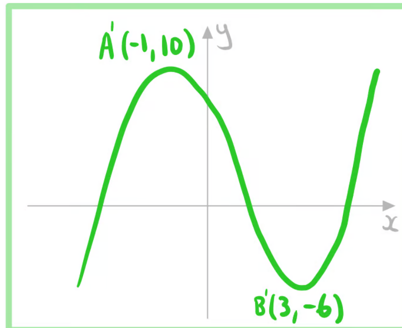
a) Sketch the graph of $y = 2f(x)$.

$y = kf(x)$ vertical stretch scale factor k

Stretch $y=f(x)$ vertically scale factor 2

A becomes $(-1, 10)$

B becomes $(3, -6)$



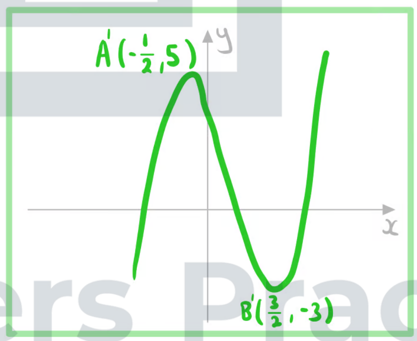
b) Sketch the graph of $y = f(2x)$.

$y = f(kx)$ horizontal stretch scale factor $\frac{1}{k}$

Stretch $y=f(x)$ horizontally scale factor $\frac{1}{2}$

A becomes $(-\frac{1}{2}, 5)$

B becomes $(\frac{3}{2}, -3)$



2.5.4 Composite Transformations of Graphs

Composite Transformations of Graphs

What transformations do I need to know?

- $y = f(x+k)$ is **horizontal translation** by vector $\begin{pmatrix} -k \\ 0 \end{pmatrix}$
  - If k is **positive** then the graph moves **left**
  - If k is **negative** then the graph moves **right**
- $y = f(x) + k$ is **vertical translation** by vector $\begin{pmatrix} 0 \\ k \end{pmatrix}$
  - If k is **positive** then the graph moves **up**
  - If k is **negative** then the graph moves **down**
- $y = f(kx)$ is a **horizontal stretch** by scale factor $\frac{1}{k}$ centred about the y-axis
  - If $k > 1$ then the graph gets **closer** to the y-axis
  - If $0 < k < 1$ then the graph gets **further** from the y-axis
- $y = kf(x)$ is a **vertical stretch** by scale factor k centred about the x-axis
  - If $k > 1$ then the graph gets **further** from the x-axis
  - If $0 < k < 1$ then the graph gets **closer** to the x-axis
- $y = f(-x)$ is a **horizontal reflection** about the y-axis
  - A **horizontal reflection** can be viewed as a special case of a **horizontal stretch**
- $y = -f(x)$ is a **vertical reflection** about the x-axis
  - A **vertical reflection** can be viewed as a special case of a **vertical stretch**

How do horizontal and vertical transformations affect each other?

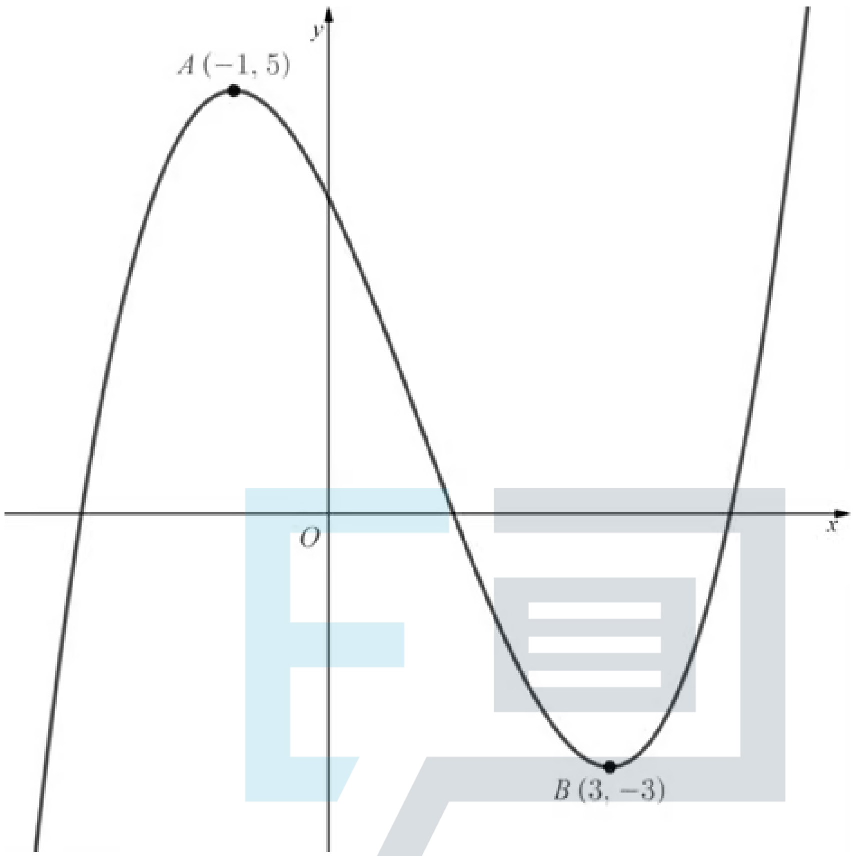
- **Horizontal and vertical transformations** are **independent** of each other
  - The horizontal transformations involved will need to be applied in their correct order
  - The vertical transformations involved will need to be applied in their correct order
- Suppose there are **two horizontal** transformation **H_1 then H_2** and **two vertical** transformations **V_1 then V_2** then they can be applied in the following orders:
  - Horizontal then vertical:
    - $H_1 H_2 V_1 V_2$
  - Vertical then horizontal:
    - $V_1 V_2 H_1 H_2$
  - Mixed up (provided that **H_1 comes before H_2** and **V_1 comes before V2**):
    - $H_1 V_1 H_2 V_2$
    - $H_1 V_1 V_2 H_2$
    - $V_1 H_1 V_2 H_2$
    - $V_1 H_1 H_2 V_2$

Exam Tip

- In an exam you are more likely to get the correct solution if you deal with one transformation at a time and sketch the graph after each transformation

Worked example

The diagram below shows the graph of $y = f(x)$.



Sketch the graph of $y = \frac{1}{2} f\left(\frac{x}{2}\right)$.

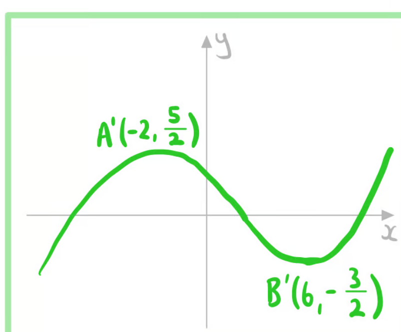
A vertical and horizontal transformation can be done in any order

$y = \frac{1}{2}f(x)$: vertical stretch scale factor $\frac{1}{2}$

$y = f\left(\frac{x}{2}\right)$: horizontal stretch scale factor 2

A becomes $\left(-2, \frac{5}{2}\right)$

B becomes $\left(6, -\frac{3}{2}\right)$



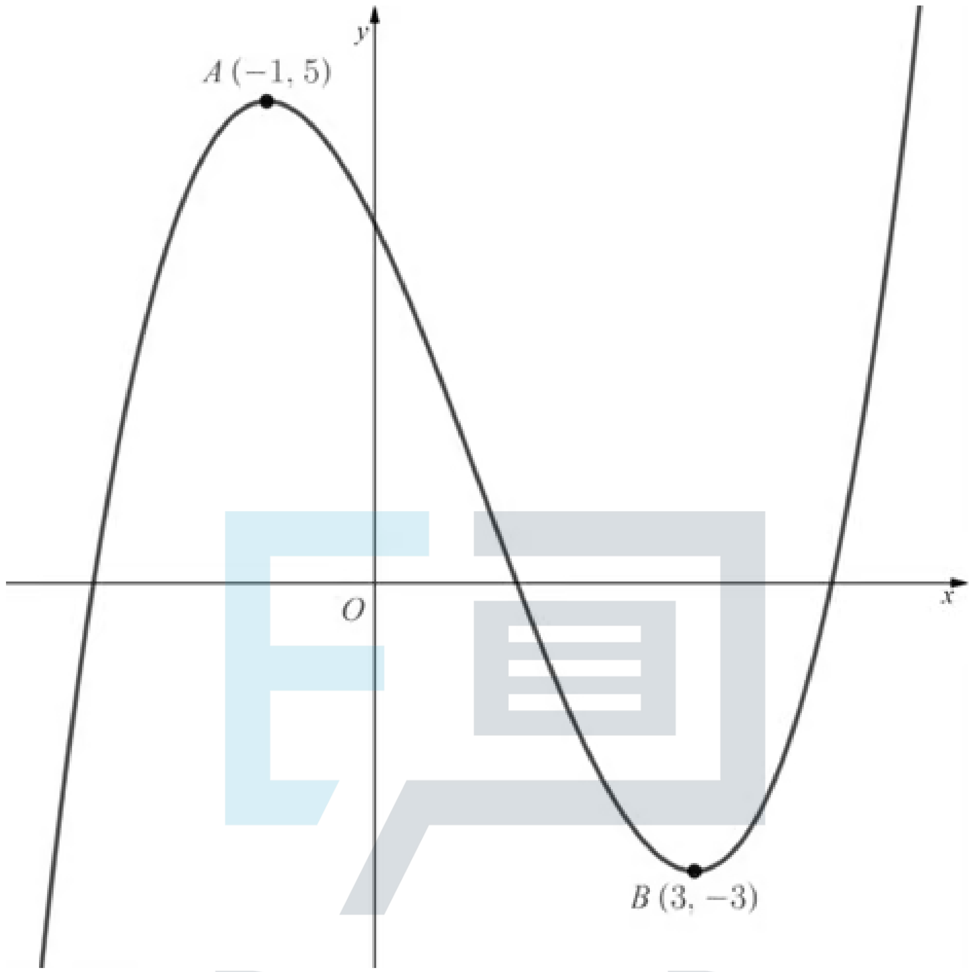
Composite Vertical Transformations af(x)+b

How do I deal with multiple vertical transformations?

- **Order matters** when you have **more than one vertical transformations**
- If you are asked to find the equation then **build up the equation** by looking at the transformations in order
    - A **vertical stretch** by scale factor *a* followed by a **translation** of $\begin{pmatrix} 0 \\ b \end{pmatrix}$
        - Stretch: $y = af(x)$
        - Then translation: $y = [af(x)] + b$
        - Final equation: $y = af(x) + b$
    - A **translation** of $\begin{pmatrix} 0 \\ b \end{pmatrix}$ followed by a **vertical stretch** by scale factor *a*
        - Translation: $y = f(x) + b$
        - Then stretch: $y = a[f(x) + b]$
        - Final equation: $y = af(x) + ab$
- If you are asked to determine the **order**
    - The order of vertical transformations **follows the order of operations**
    - First write the equation in the form $y = af(x) + b$
        - **First stretch vertically** by scale factor *a*
        - If *a* is negative then the **reflection and stretch** can be **done in any order**
        - **Then translate** by $\begin{pmatrix} 0 \\ b \end{pmatrix}$

Worked example

The diagram below shows the graph of $y = f(x)$.



Sketch the graph of $y = 3f(x) - 2$.

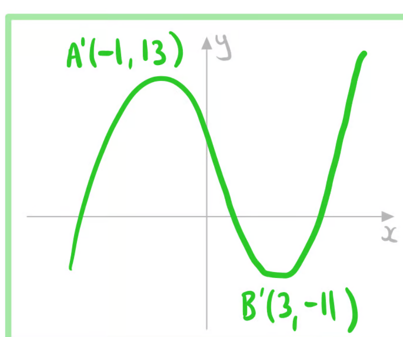
The order vertical transformations follows the order of operations

$y = 3f(x)$: Vertical stretch scale factor 3

$y = f(x) - 2$: Translate $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

A becomes $(-1, 13)$

B becomes $(3, -11)$



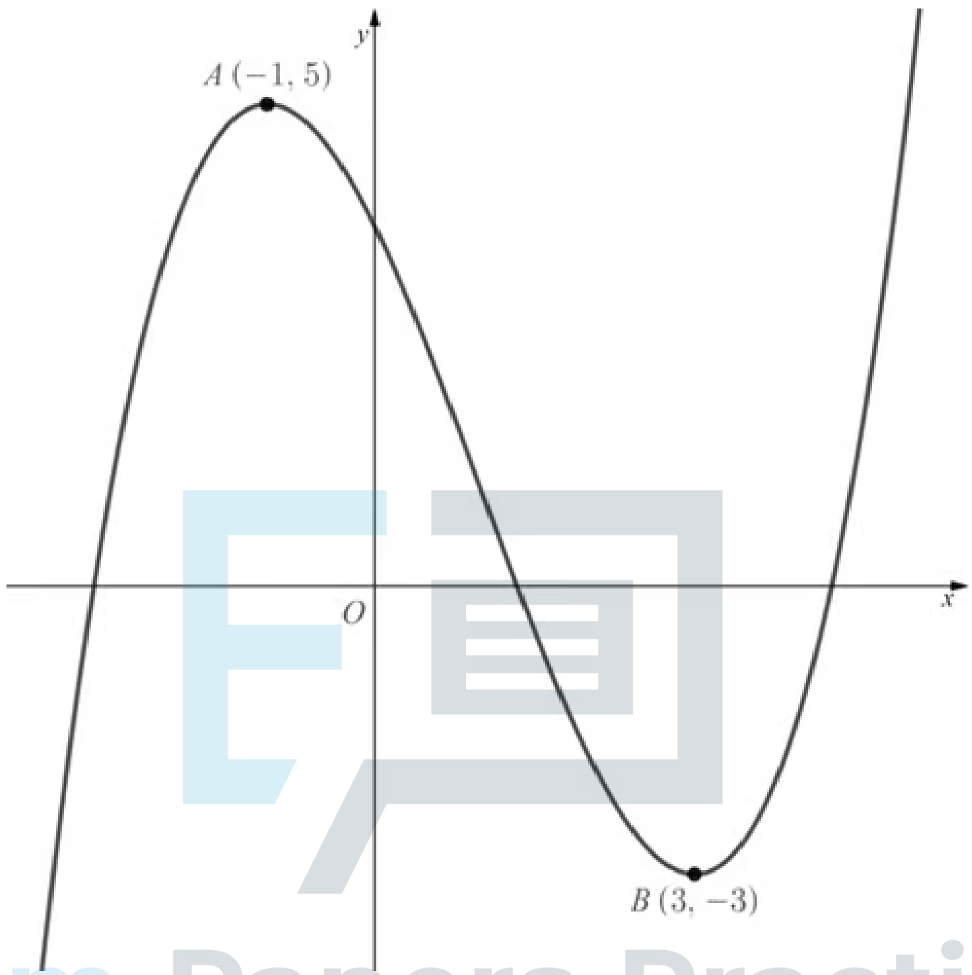
Composite Horizontal Transformations f(ax+b)

How do I deal with multiple horizontal transformations?

- **Order matters** when you have **more than one horizontal transformations**
- If you are asked to find the equation then **build up the equation** by looking at the transformations in order
  - A **horizontal stretch** by scale factor $\frac{1}{a}$ followed by a **translation** of $\begin{pmatrix} -b \\ 0 \end{pmatrix}$
    - Stretch: $y = f(ax)$
    - Then translation: $y = f(a(x+b))$
    - Final equation: $y = f(ax+ab)$
  - A **translation** of $\begin{pmatrix} -b \\ 0 \end{pmatrix}$ followed by a **horizontal stretch** by scale factor $\frac{1}{a}$
    - Translation: $y = f(x+b)$
    - Then stretch: $y = f((ax)+b)$
    - Final equation: $y = f(ax+b)$
- If you are asked to determine the **order**
  - First write the equation in the form $y = f(ax+b)$
  - The order of horizontal transformations **is the reverse of the order of operations**
    - **First translate** by $\begin{pmatrix} -b \\ 0 \end{pmatrix}$
    - **Then stretch** by scale factor $\frac{1}{a}$
    - If *a* is negative then the **reflection and stretch** can be **done in any order**

Worked example

The diagram below shows the graph of $y = f(x)$.



Sketch the graph of $y = f(2x - 1)$.

The order of horizontal transformations is the reverse of the order of operations

$y = f(x-1)$: Translate $\begin{pmatrix}1\\0\end{pmatrix}$

$y = f(2x)$: Horizontal stretch scale factor $\frac{1}{2}$

A becomes $(0, 5)$

B becomes $(2, -3)$