Exam Papers Practice

2.4 Further Functions & Graphs

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Course	DP IB Maths
Section	2. Functions
Topic	2.4 Further Functions & Graphs
Difficulty	Medium

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To be used by all students preparing for DP IB Maths AA SL
Students of other boards may also find this useful

Question 1

a) i) y-intercepts occur when $x = 0$.

Sub $x = 0$ into $f(x)$.

$f(0) = -(0)^5 + 2020$

$f(0) = 2020$

Hence the y-intercept for f is $(0, 2020)$.

ii) Sub $x = 0$ into $g(x)$.

$$g(0) = \frac{1}{\sqrt{(1-(0))^3}} - 2$$

$g(0) = -1$

Hence the y-intercept for g is $(0, -1)$.

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b)i) x-intercepts occur when the function equals zero.

Set $f(x) = 0$ and solve for x on your GDC.

$-x^5 + 2020 = 0$

$x \approx 4.58$

Hence the x-intercept for f is $(4.58, 0)$.

ii) Set $g(x) = 0$ and solve for x on your GDC.

$\frac{1}{\sqrt{(1-x)^3}} - 2 = 0$

$x \approx 0.370$

Hence the x-intercept for g is $(0.37, 0)$.

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c)i) The vertical asymptote is when the denominator of $g(x)$ equals zero.

[denominator of g] $= 0$

$\sqrt{(1-x)^3} = 0$

$x = 1$

Hence the equation of the vertical asymptote is $x = 1$

ii) As x tends towards negative infinity $(-\infty)$, $\frac{1}{\sqrt{(1-x)^3}}$ tends towards zero.

$$g(x) = \frac{1}{\sqrt{(1-x)^3}} - 2$$

$$\lim_{x \to -\infty} g(x) = 0 - 2$$
$$= -2$$

Hence the equation of the horizontal asymptote is $y = -2$.

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Question 2

a) i) $f(x)$ is undefined when the denominator $= 0$.

$x - 4 = 0$

$x = 4$

Vertical asymptote: $x = 4$

ii) As x tends towards $\pm\infty$

$f(x)$ tends towards 2.

$$\lim_{x \to \pm\infty} f(x) = \frac{2(\pm\infty) + 1}{(\pm\infty) - 4}$$

Horizontal asymptote: $y = 2$

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b) Find $f^{-1}(x)$.

$y = f(x)$

$y = \frac{2x+1}{x-4}$ ⟩ swap x and y

$x = \frac{2y+1}{y-4}$

$x(y-4) = 2y+1$

$xy - 4x = 2y + 1$

$xy - 2y = 4x + 1$

$y(x-2) = 4x + 1$

$y = \frac{4x+1}{x-2}$

$f^{-1}(x) = \frac{4x+1}{x-2}$

c) $f(x)$ is undefined when the denominator = 0.

$x - 2 = 0$

$x = 2$

Vertical asymptote: $x = 2$

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Question 3

a) x-intercepts occur when $f(x) = 0$.

$0 = \ln(x + 2)$

$1 = x + 2$

$x = -1$

x-intercept at $(-1, 0)$.

y-intercepts occur when $x = 0$.

$f(0) = \ln(0 + 2)$

$f(0) = 0.6931...$

$= 0.693$ (3sf)

y-intercept at $(0, 0.693)$.

b) $f(x)$ is undefined when $x + 2 < 0$.

$x + 2 < 0$

Vertical asymptote: $x = -2$

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c) Find $f^{-1}(x)$.

$y = f(x)$

$y = \ln(x + 2)$ ↓ swap x and y

$x = \ln(y + 2)$

$y = e^x - 2$

$f^{-1}(x) = e^x - 2$

$f(x) = f^{-1}(x)$

$\ln(x + 2) = e^x - 2$

Graph $f(x)$ and $f^{-1}(x)$ on your GDC and find their intersection.

Intersection:

$(-1.84, -1.84)$ and $(1.14, 1.14)$

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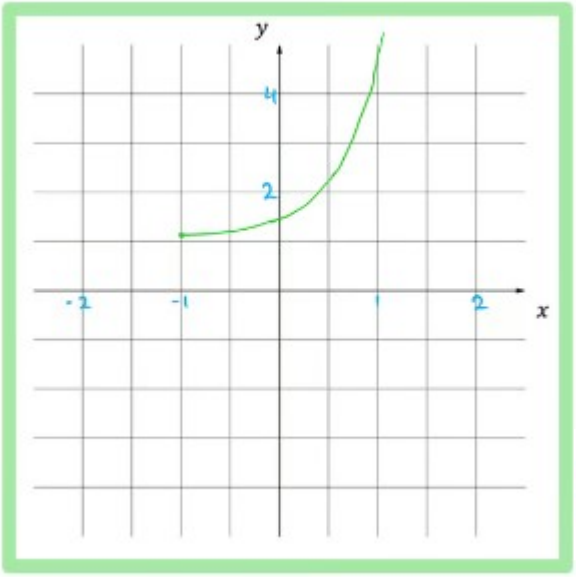
Question 4



a) Graph $f(x)$ on your GDC.

b) Find $f^{-1}(x)$

$y = f(x)$

$y = 0.5e^{2x} + 1$

$x = 0.5e^{2y} + 1$

$\frac{x-1}{0.5} = e^{2y}$

$2y = \ln 2(x-1)$

$y = \frac{1}{2}\ln 2(x-1)$

$\therefore A = \frac{1}{2} \quad b = 2 \quad c = 1$

Question 5

a) $t=0 \quad m=100$

$100 = m_0 e^{-k(0)} = m_0$

$m_0 = 100$

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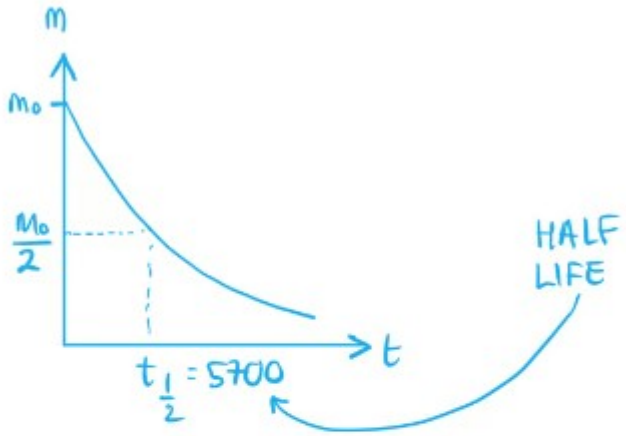
b)



Since half life is 5700y, this is the time it takes for the initial mass m_0 (100g) to half to 50g.

c) Sub values from (b) into model

$50 = 100e^{-k5700}$

$\frac{1}{2} = e^{-k5700}$

$\ln\frac{1}{2} = \ln e^{-5700k}$

$\ln\frac{1}{2} = -5700k$

$k = \frac{-\ln\frac{1}{2}}{5700} = 1.22\times10^{-4}$ (3sf)

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d) Sub $m_0 = 60$ and $t = 2000$ into model

$$m = 60e^{-1.22\times10^{-4}(2000)}$$

$$\boxed{m = 47.0\text{g}} \quad (3\text{sf})$$

Question 6

a) $P = 2500 \quad y = 1$ $\qquad 2500 = P_0(1)^k$

$P = 3700 \quad y = 2$ $\qquad 3700 = P_0(2)^k$

b) dividing ① by ②

$$\frac{2500}{3700} = \left(\frac{1}{2}\right)^k$$

$$k = \log_{\frac{1}{2}}\left(\frac{2500}{3700}\right) = 0.566 = k \quad (3\text{sf})$$

sub $k = 0.566$ into ①

$$2500 = P_0(1)^{0.566} = P_0$$

$$\boxed{P_0 = £2500}$$

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c) Using P_0 and k from (b)

$P = 2500y^{0.566}$

at $y = 3$ $P = 2500(3)^{0.566} =$ £4653.70 (to the nearest penny)

$y = 4$ $P = 2500(4)^{0.566} =$ £5476 (exactly!)

d) $\log P = \log P_0 y^k$

rewrite as two terms

$= \log P_0 + \log y^k$

rewrite power as coefficient

$\log P = \log P_0 + k \log y$

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Question 7

a) $t = 0$

$B = 16e^{0.85(0)} = 16$

b) $t = 3$

$B = 16e^{0.85(3)} = 205$ (3sf)

c) Sub B=500 into equation

$500 = 16e^{0.85t}$

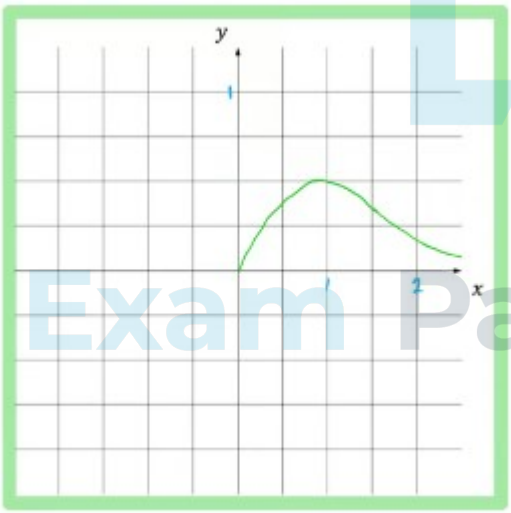
$\frac{125}{4} = e^{0.85t}$

$\ln\frac{125}{4} = \ln e^{0.85t}$

$= 0.85t \ln e^{1}$

$t = \frac{1}{0.85}\ln\frac{125}{4} = 4.05y$ (3sf)

Question 8



a) Graph $V_1(t)$ on your GDC.

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b) Find the maximum of $V_1(t)$ on your GDC.

maximum: (0.8, 0.5)

0.8×24 hours $= 19.2$ hours

19.2 hours = 19 hrs and 12 mins

9 am + 19 hrs and 12 mins = 4:12 am

4:12 am on Tuesday

c) Maximum: (0.8, 0.5)

Find t when $V(t) = 0.25$

$0.25 = 1.7te^{-1.25t}$

$t = 2.14348$

$\therefore$ days $= 2.14348 - 0.8$
$= 1.34348$

$0.34348 \times 24 = 8.24352$

$0.24352 \times 60 = 14.6 \approx 15$

1 day 8hrs and 15mins

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d) Maximum : (0.8, 0.5)

Find t when $V(t) = 0.005$

$0.005 = 1.7te^{-1.25t}$

$t = 6.11126$ days

$0.11126 \times 24 = 2.67024$

$0.67024 \times 60 = 40.2 \approx 40$ mins

t = 6 days + 2hrs + 40 mins

11:40 am on Sunday.

e) Graph $V_2(t)$ and find its maximum

maximum : (0.769, 0.5)

time = 0.8 − 0.769
= 0.031 days

$0.031 \times 24 = 0.744$ hrs

$0.744 \times 60 = 44.64$ mins

45 minutes

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Question 9

a) i) x-intercepts occur when $f(x) = 0$.

$$0 = -\frac{3}{x-3} \qquad (f(x) \neq 0)$$

No solutions, $f(x)$ does not cross the x-axis.

ii) y-intercepts occur when $x = 0$.

$$f(0) = -\frac{3}{(0)-3}$$

$$f(0) = 1$$

y-intercept at $(0, 1)$.

iii) Range = $(-\infty, 0) \cup (0, \infty)$

$(f(x) \neq 0)$

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b) Find $f^{-1}(x)$.

$y = f(x)$

$y = -\frac{3}{x-3}$ → swap x and y

$x = -\frac{3}{y-3}$ → rearrange

$y = -\frac{3}{x} + 3$

$\therefore f^{-1}(x) = -\frac{3}{x} + 3$

Sub $x = -1$ into $f^{-1}(x)$.

$f^{-1}(-1) = -\frac{3}{(-1)} + 3$

$f^{-1}(-1) = 3 + 3$

$f^{-1}(-1) = 6$

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c) $g(x) = f(x+3) + 1$

Sub in $(x+3)$ for x and $+1$.

$$g(x) = -\frac{3}{(x+3)-3} + 1$$

$$g(x) = -\frac{3}{x} + 1$$

For $g(x)$, $x \neq 0$.

Domain = $(-\infty, 0) \cup (0, \infty)$

$$\lim_{x \to \pm\infty} g(x) = -\frac{3}{(\pm\infty)} + 1$$

$$\lim_{x \to \pm\infty} g(x) = 0 + 1$$

Range = $(-\infty, 1) \cup (1, \infty)$

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