DP IB Maths: AA HL

2.4 Other Functions & Graphs

Contents

2.4.1 Exponential & Logarithmic Functions

Exponential Functions & Graphs

What is an exponential function?

- An **exponential function** is defined by $f(x) = a^x,\ a > 0$
- Its **domain** is the set of **all real values**
- Its **range** is the set of **all positive real values**
- An important exponential function is $f(x) = \mathrm{e}^x$
  - Where e is the mathematical constant 2.718...
- Any exponential function can be written using e
  - $a^x = \mathrm{e}^{x \ln a}$
    - This is given in the **formula booklet**

What are the key features of exponential graphs?

- The graphs have a ***y*-intercept** at (0, 1)
- The graph **will always** pass through the **point** (1, *a*)
- The graphs **do not have any roots**
- The graphs have **a horizontal asymptote** at the *x*-axis: $y = 0$
  - For ***a* > 1** this is the **limiting value** when *x* tends to **negative infinity**
  - For **0 < *a* < 1** this is the **limiting value** when *x* tends to **positive infinity**
- The graphs **do not have any minimum or maximum points**

Logarithmic Functions & Graphs

What is a logarithmic function?

- A **logarithmic function** is of the form $f(x) = \log_a x,\ x > 0$
- Its **domain** is the set of all **positive real values**
  - You can't take a log of zero or a negative number
- Its **range** is set of **all real values**
- $\log_a x$ and a^x are **inverse** functions
- An important logarithmic function is $f(x) = \ln x$
  - This is the natural logarithmic function $\ln x \equiv \log_e x$
  - This is the inverse of e^x
    - $\ln e^x = x$ and $e^{\ln x} = x$
- Any logarithmic function can be written using ln
  - $\log_a x = \dfrac{\ln x}{\ln a}$ using the change of base formula

What are the key features of logarithmic graphs?

- The graphs **do not have a *y*-intercept**
- The graphs have **one root** at (1, 0)
- The graphs **will always** pass through the point (*a*, 1)
- The graphs have a **vertical asymptote** at the *y*-axis: $x = 0$
- The graphs **do not have any minimum or maximum points**

Worked example

The function f is defined by $f(x) = \log_5 x$ for $x > 0$.

a) Write down the inverse of f. Give your answer in the form $e^{g(x)}$.

Formula booklet

Exponents & logarithms	$a^x = b \Leftrightarrow x = \log_a b$	$a > 0, b > 0, a \neq 1$

$x = \log_5 y \Leftrightarrow y = 5^x$

Formula booklet

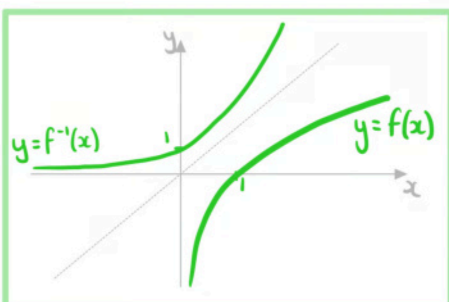
Exponential & logarithmic functions	$a^x = e^{x \ln a}$

$5^x = e^{x \ln 5}$

$f^{-1}(x) = e^{x \ln 5}$

b) Sketch the graphs of f and its inverse on the same set of axes.

f and f^{-1} are reflections in line $y = x$



2.4.2 Solving Equations

Solving Equations Analytically

How can I solve equations analytically where the unknown appears only once?

- These equations can be **solved by rearranging**
- For **one-to-one functions** you can just apply the **inverse**
  - Addition and subtraction are inverses
    - $y = x + k \Leftrightarrow x = y - k$
  - Multiplication and division are inverses
    - $y = kx \Leftrightarrow x = \frac{y}{k}$
  - Taking the reciprocal is a self-inverse
    - $y = \frac{1}{x} \Leftrightarrow x = \frac{1}{y}$
  - Odd powers and roots are inverses
    - $y = x^n \Leftrightarrow x = \sqrt[n]{y}$
    - $y = x^n \Leftrightarrow x = y^{\frac{1}{n}}$
  - Exponentials and logarithms are inverses
    - $y = a^x \Leftrightarrow x = \log_a y$
    - $y = e^x \Leftrightarrow x = \ln y$
- For **many-to-one functions** you will need to use your knowledge of the functions to find the **other solutions**
  - Even powers lead to positive and negative solutions
    - $y = x^n \Leftrightarrow x = \pm\sqrt[n]{y}$
  - Modulus functions lead to positive and negative solutions
    - $y = |x| \Leftrightarrow x = \pm y$
  - Trigonometric functions lead to infinite solutions using their symmetries
    - $y = \sin x \Leftrightarrow x = 2k\pi + \arcsin y \text{ or } x = (1 + 2k)\pi - \arcsin y$
    - $y = \cos x \Leftrightarrow x = 2k\pi \pm \arccos y$
    - $y = \tan x \Leftrightarrow x = k\pi + \arctan y$
- Take care when you apply **many-to-one functions** to **both sides** of an equation as this can create **additional solutions** which are incorrect
  - For example: squaring both sides
    - $x + 1 = 3$ has one solution $x = 2$
    - $(x + 1)^2 = 3^2$ has two solutions $x = 2$ and $x = -4$
- Always **check your solutions** by substituting back into the original equation

How can I solve equations analytically where the unknown appears more than once?

- Sometimes it is possible to **simplify expressions** to make the **unknown appear only once**
- **Collect all terms** involving *x* on **one side** and try to simplify into one term
  - For **exponents** use
    - $a^{f(x)} \times a^{g(x)} = a^{f(x)+g(x)}$
    - $\frac{a^{f(x)}}{a^{g(x)}} = a^{f(x)-g(x)}$
    - $(a^{f(x)})^{g(x)} = a^{f(x)\times g(x)}$
    - $a^{f(x)} = e^{f(x)\ln a}$
  - For **logarithms** use
    - $\log_a f(x) + \log_a g(x) = \log_a(f(x) \times g(x))$
    - $\log_a f(x) - \log_a g(x) = \log_a\left(\frac{f(x)}{g(x)}\right)$
    - $n\log_a f(x) = \log_a(f(x))^n$

How can I solve equations analytically when the equation can't be simplified?

- Sometimes it is **not possible to simplify** equations
- Most of these equations **cannot be solved analytically**
- A **special case** that can be solved is where the equation can be **transformed into a quadratic** using a substitution
  - These will have **three terms** and involve the same type of function
- **Identify** the **suitable substitution** by considering which **function is a square of another**
  - For example: the following can be transformed into $2y^2 + 3y - 4 = 0$
    - $2x^4 + 3x^2 - 4 = 0$ using $y = x^2$
    - $2x + 3\sqrt{x} - 4 = 0$ using $y = \sqrt{x}$
    - $\frac{2}{x^6} + \frac{3}{x^3} - 4 = 0$ using $y = \frac{1}{x^3}$
    - $2e^{2x} + 3e^x - 4 = 0$ using $y = e^x$
    - $2 \times 25^x + 3 \times 5^x - 4 = 0$ using $y = 5^x$
    - $2^{2x+1} + 3 \times 2^x - 4 = 0$ using $y = 2^x$
    - $2(x^3 - 1)^2 + 3(x^3 - 1) - 4 = 0$ using $y = x^3 - 1$
- To **solve**:
  - Make the **substitution** $y = f(x)$
  - **Solve** the **quadratic equation** $ay^2 + by + c = 0$ to get y_1 & y_2
  - **Solve** $f(x) = y_1$ and $f(x) = y_2$
    - Note that some equations might have **zero or several solutions**

Can I divide both sides of an equation by an expression?

- When dividing by an expression you must consider whether the **expression could be zero**
- Dividing by an expression that could be zero could result in you **losing solutions to the original equation**
  - For example: $(x+1)(2x-1)=3(x+1)$
    - If you divide both sides by $(x+1)$ you get $2x-1=3$ which gives $x=2$
    - However $x=-1$ is also a solution to the original equation
- To ensure you **do not lose solutions** you can:
  - **Split the equation into two equations**
    - One where the dividing expression equals zero: $x+1=0$
    - One where the equation has been divided by the expression: $2x-1=3$
  - **Make the equation equal zero and factorise**
    - $(x+1)(2x-1)-3(x+1)=0$
    - $(x+1)(2x-1-3)=0$ which gives $(x+1)(2x-4)=0$
    - Set each factor equal to zero and solve: $x+1=0$ and $2x-4=0$

Worked example

Find the exact solutions for the following equations:

a) $5 - 2\log_4 x = 0$.

Rearrange using inverse functions

$5 - 2\log_4 x = 0$

$2\log_4 x = 5$ — $y = x - k \Leftrightarrow x = y + k$

$\log_4 x = \frac{5}{2}$ — $y = kx \Leftrightarrow x = \frac{y}{k}$

$x = 4^{5/2}$ — $y = \log_a x \Leftrightarrow x = a^y$

$x = (\sqrt{4})^5$ — $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$

$x = 32$

b) $x = \sqrt{x+2}$.

Square both sides (Many-to-one function)

$x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0$

$(x-2)(x+1) = 0 \Rightarrow x = 2$ or $x = -1$

Check whether each solution is valid

$x = 2$: LHS $= 2$ RHS $= \sqrt{2+2} = 2$ ✓

$x = -1$: LHS $= -1$ RHS $= \sqrt{-1+2} = 1$ ✗

$x = 2$

c) $e^{2x} - 4e^x - 5 = 0$.

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Notice $e^{2x} = (e^x)^2$, let $y = e^x$

$y^2 - 4y - 5 = 0 \Rightarrow (y+1)(y-5) = 0$

$y = -1$ or $y = 5$

Solve using $y = e^x$

$e^x = -1$ has no solutions as $e^x > 0$

$e^x = 5 \quad \therefore \quad x = \ln 5$

$\boxed{x = \ln 5}$

Solving Equations Graphically

How can I solve equations graphically?

- To solve $f(x) = g(x)$
  - One method is to **draw the graphs** $y = f(x)$ and $y = g(x)$
    - The **solutions** are the **x-coordinates** of the points of **intersection**
  - Another method is to **draw the graph** $y = f(x) - g(x)$ or $y = g(x) - f(x)$
    - The **solutions** are the **roots (zeros)** of this graph
      - This method is sometimes quicker as it involves **drawing only one graph**

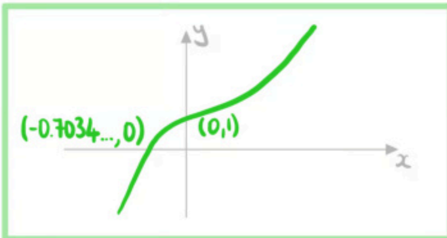
Why do I need to solve equations graphically?

- Some equations **cannot be solved analytically**
  - **Polynomials** of degree higher than 4
    - $x^5 - x + 1 = 0$
  - Equations involving **different types of functions**
    - $e^x = x^2$

Worked example

a) Sketch the graph $y = e^x - x^2$.

Sketch using GDC



b) Hence find the solution to $e^x = x^2$.

$e^x = x^2$ when $e^x - x^2 = 0$

Solution is the x-intercept of $y = e^x - x^2$

$x = -0.703$ (3sf)

2.4.3 Modelling with Functions

Modelling with Functions

What is a mathematical model?

- A **mathematical model** simplifies a real-world situation so it can be described using mathematics
  - The model can then be used to make predictions
- **Assumptions** about the situation are made in order to simplify the mathematics
- Models can be **refined** (improved) if further information is available or if the model is compared to real-world data

How do I set up the model?

- The question could:
  - give you the equation of the model
  - tell you about the relationship
    - It might say the relationship is linear, quadratic, etc
  - ask you to suggest a **suitable model**
    - Use your knowledge of each model
    - E.g. if it is compound interest then an exponential model is the most appropriate
- You may have to determine a **reasonable domain**
  - Consider real-life context
    - E.g. if dealing with hours in a day then
    - E.g. if dealing with physical quantities (such as length) then
  - Consider the **possible ranges**
    - If the outcome cannot be negative then you want to choose a domain which corresponds to a range with no negative values
    - **Sketching the graph** is helpful to determine a suitable domain

Which models might I need to use?

- You could be given any model and be expected to use it
- Common models include:
  - **Linear**
    - Arithmetic sequences
    - Linear regression
  - **Quadratic**
    - Projectile motion
    - The height of a cable supporting a bridge
    - Profit
  - **Exponential**
    - Geometric sequences
    - Exponential growth and decay
    - Compound interest

- **Logarithmic**
  - Richter scale for the magnitude of earthquakes
- **Rational**
  - Temperature of a cup of coffee
- **Trigonometric**
  - The depth of a tide

How do I use a model?

- You can use a model by substituting in values for the variable to **estimate outputs**
  - For example: Let $h(t)$ be the height of a football t seconds after being kicked
    - $h(3)$ will be an estimate for the height of the ball 3 seconds after being kicked
- Given an **output** you can **form an equation** with the model to **estimate the input**
  - For example: Let $P(n)$ be the profit made by selling n items
    - Solving $P(n) = 100$ will give you an estimate for the number of items needing to be sold to make a profit of 100
- If your variable is **time** then substituting $t = 0$ will give you the **initial value** according to the model
- Fully understand the **units for the variables**
  - If the units of P are measured in **thousand dollars** then $P = 3$ represents $3000
- Look out for **key words** such as:
  - Initially
  - Minimum/maximum
  - Limiting value

What do I do if some of the parameters are unknown?

- A general method is to **form equations** by substituting in given values
  - You can form **multiple equations** and **solve them simultaneously** using your GDC
  - This method **works for all models**
- The **initial value** is the value of the function when the variable is 0
  - This is **normally one of the parameters** in the equation of the model

Worked example

The temperature, T°C, of a cup of coffee is monitored. Initially the temperature is 80°C and 5 minutes later it is 40°C . It is suggested that the temperature follows the model:

$$T(t) = Ae^{kt} + 16, \ t \geq 0.$$

where t is the time, in minutes, after the coffee has been made.

a) State the value of A.

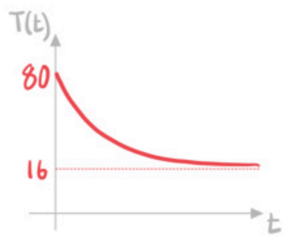
Initially temperature is 80°C

$T(0) = 80$

$Ae^{-k(0)} + 16 = 80$

$A + 16 = 80$

$A = 64$



b) Find the exact value of k.

$t = 5$, $T = 40$

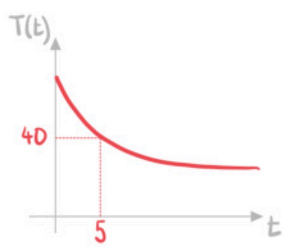
$40 = 64e^{5k} + 16$

$64e^{5k} = 24$

$e^{5k} = \frac{3}{8}$

$5k = \ln \frac{3}{8}$

$k = \frac{1}{5} \ln \frac{3}{8}$



c) Find the time taken for the temperature of the coffee to reach 30°C.

Find t such that $T(t) = 30$

$30 = 64e^{kt} + 16$

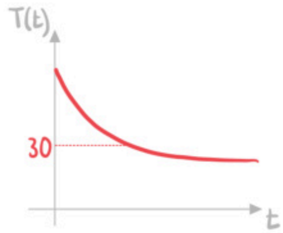
Leave as k until the end to save writing $\frac{1}{5}\ln\frac{3}{8}$ each time

$64e^{kt} = 14$

$e^{kt} = \frac{7}{32}$

$kt = \ln\frac{7}{32}$

$t = \frac{\ln\frac{7}{32}}{k} = \frac{\ln\frac{7}{32}}{\frac{1}{5}\ln\frac{3}{8}} = 7.7476...$



7.75 minutes (3sf)