DP IB Maths: AA SL

2.4 Further Functions & Graphs

Contents

2.4.1 Reciprocal & Rational Functions

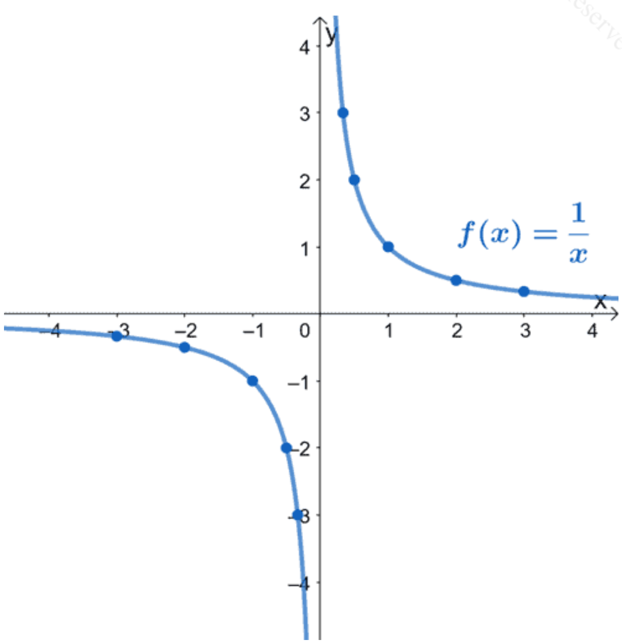
Reciprocal Functions & Graphs

What is the reciprocal function?

- The **reciprocal function** is defined by $f(x) = \frac{1}{x},\ x \neq 0$
- Its **domain** is the set of **all real values except 0**
- Its **range** is the set of **all real values except 0**
- The reciprocal function has a **self-inverse** nature
  - $f^{-1}(x) = f(x)$
  - $(f \circ f)(x) = x$

What are the key features of the reciprocal graph?

- The graph **does not have a *y*-intercept**
- The graph **does not have any roots**
- The graph has **two asymptotes**
  - A **horizontal** asymptote at the *x*-axis: $y = 0$
    - This is the **limiting value** when the absolute value of *x* gets very large
  - A **vertical** asymptote at the *y*-axis: $x = 0$
    - This is the value that causes the **denominator to be zero**
- The graph has **two axes of symmetry**
  - $y = x$
  - $y = -x$
- The graph **does not have any minimum or maximum points**



Linear Rational Functions & Graphs

What is a rational function?

- A **rational function** is of the form $f(x) = \dfrac{ax+b}{cx+d},\ x \neq -\dfrac{d}{c}$
- Its **domain** is the set of **all real values except** $-\dfrac{d}{c}$
- Its **range** is the set of **all real values except** $\dfrac{a}{c}$
- The **reciprocal function** is a **special case** of a rational function

What are the key features of rational graphs?

- The graph has a ***y*-intercept** at $\left(0,\ \dfrac{b}{d}\right)$ provided $d \neq 0$
- The graph has **one root** at $\left(-\dfrac{b}{a},\ 0\right)$ provided $a \neq 0$
- The graph has **two asymptotes**
  - A **horizontal** asymptote: $y = \dfrac{a}{c}$
    - This is the **limiting value** when the absolute value of *x* gets very large
  - A **vertical** asymptote: $x = -\dfrac{d}{c}$
    - This is the value that causes the **denominator to be zero**
- The graph **does not have any minimum or maximum points**
- If you are asked to **sketch or draw** a rational graph:
  - Give the **coordinates** of any **intercepts** with the axes
  - Give the **equations** of the **asymptotes**

Worked example

The function f is defined by $f(x) = \dfrac{10-5x}{x+2}$ for $x \neq -2$.

a) Write down the equation of

(i) the vertical asymptote of the graph of f,

(ii) the horizontal asymptote of the graph of f.

(i) Vertical asymptote is when denominator equals zero

$x+2=0$ $\boxed{x = -2}$

(ii) Horizontal asymptote is limiting value as x gets large

$\lim_{x\to\infty} \frac{10-5x}{x+2} = \lim_{x\to\infty} \frac{-5\cancel{x}}{\cancel{x}}$ $\boxed{y = -5}$

b) Find the coordinates of the intercepts of the graph of f with the axes.

y-intercept occurs when $x=0$

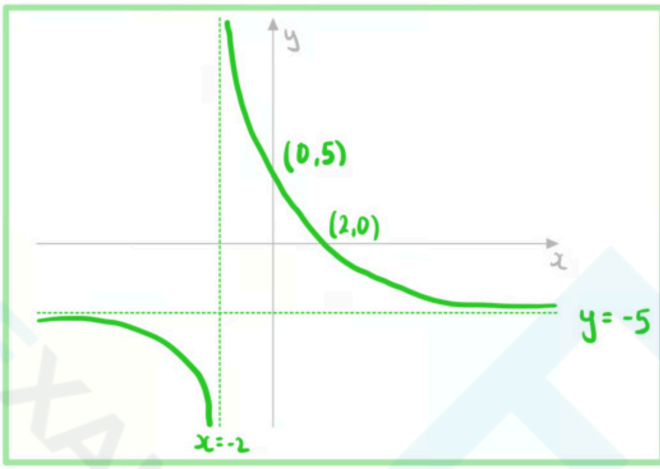
$y = \frac{10-5(0)}{0+2} = 5$ $\boxed{(0,5)}$

x-intercept occurs when $y=0$

$\frac{10-5x}{x+2} = 0 \Rightarrow 10-5x=0 \Rightarrow x=2$ $\boxed{(2,0)}$

c) Sketch the graph of f.

Include asymptotes and intercepts



2.4.2 Exponential & Logarithmic Functions

Exponential Functions & Graphs

What is an exponential function?

- An **exponential function** is defined by $f(x) = a^x,\ a > 0$
- Its **domain** is the set of **all real values**
- Its **range** is the set of **all positive real values**
- An important exponential function is $f(x) = e^x$
  - Where e is the mathematical constant 2.718...
- Any exponential function can be written using e
  - $a^x = e^{x \ln a}$
    - This is given in the **formula booklet**

What are the key features of exponential graphs?

- The graphs have a ***y*-intercept** at (0, 1)
- The graph **will always** pass through the **point** (1, *a*)
- The graphs **do not have any roots**
- The graphs have **a horizontal asymptote** at the *x*-axis: $y = 0$
  - For ***a* > 1** this is the **limiting value** when *x* tends to **negative infinity**
  - For **0 < *a* < 1** this is the **limiting value** when *x* tends to **positive infinity**
- The graphs **do not have any minimum or maximum points**

Logarithmic Functions & Graphs

What is a logarithmic function?

- A **logarithmic function** is of the form $f(x) = \log_a x,\ x > 0$
- Its **domain** is the set of all **positive real values**
  - You can't take a log of zero or a negative number
- Its **range** is set of **all real values**
- $\log_a x$ and a^x are **inverse** functions
- An important logarithmic function is $f(x) = \ln x$
  - This is the natural logarithmic function $\ln x \equiv \log_e x$
  - This is the inverse of e^x
    - $\ln e^x = x$ and $e^{\ln x} = x$
- Any logarithmic function can be written using ln
  - $\log_a x = \dfrac{\ln x}{\ln a}$ using the change of base formula

What are the key features of logarithmic graphs?

- The graphs **do not have a *y*-intercept**
- The graphs have **one root** at (1, 0)
- The graphs **will always** pass through the point (*a*, 1)
- The graphs have a **vertical asymptote** at the *y*-axis: $x = 0$
- The graphs **do not have any minimum or maximum points**

Worked example

The function f is defined by $f(x) = \log_5 x$ for $x > 0$.

a) Write down the inverse of f. Give your answer in the form $e^{g(x)}$.

Formula booklet

Exponents & logarithms	$a^x = b \Leftrightarrow x = \log_a b$	$a > 0, b > 0, a \neq 1$

$x = \log_5 y \Leftrightarrow y = 5^x$

Formula booklet

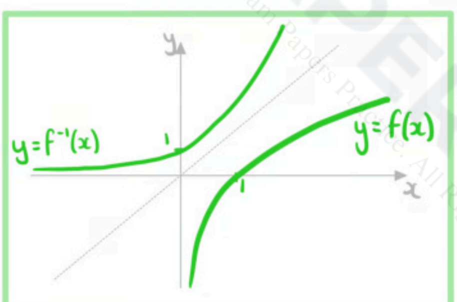
Exponential & logarithmic functions	$a^x = e^{x \ln a}$

$5^x = e^{x \ln 5}$

$\boxed{f^{-1}(x) = e^{x \ln 5}}$

b) Sketch the graphs of f and its inverse on the same set of axes.

f and f^{-1} are reflections in line $y = x$



2.4.3 Solving Equations

Solving Equations Analytically

How can I solve equations analytically where the unknown appears only once?

- These equations can be **solved by rearranging**
- For **one-to-one functions** you can just apply the **inverse**
  - Addition and subtraction are inverses
    - $y = x + k \Leftrightarrow x = y - k$
  - Multiplication and division are inverses
    - $y = kx \Leftrightarrow x = \dfrac{y}{k}$
  - Taking the reciprocal is a self-inverse
    - $y = \dfrac{1}{x} \Leftrightarrow x = \dfrac{1}{y}$
  - Odd powers and roots are inverses
    - $y = x^n \Leftrightarrow x = \sqrt[n]{y}$
    - $y = x^n \Leftrightarrow x = y^{\frac{1}{n}}$
  - Exponentials and logarithms are inverses
    - $y = a^x \Leftrightarrow x = \log_a y$
    - $y = \mathrm{e}^x \Leftrightarrow x = \ln y$
- For **many-to-one functions** you will need to use your knowledge of the functions to find the **other solutions**
  - Even powers lead to positive and negative solutions
    - $y = x^n \Leftrightarrow x = \pm\sqrt[n]{y}$
  - Modulus functions lead to positive and negative solutions
    - $y = |x| \Leftrightarrow x = \pm y$
  - Trigonometric functions lead to infinite solutions using their symmetries
    - $y = \sin x \Leftrightarrow x = 2k\pi + \sin^{-1} y \text{ or } x = (1 + 2k)\pi - \sin^{-1} y$
    - $y = \cos x \Leftrightarrow x = 2k\pi \pm \cos^{-1} y$
    - $y = \tan x \Leftrightarrow x = k\pi + \tan^{-1} y$
- Take care when you apply **many-to-one functions** to **both sides** of an equation as this can create **additional solutions** which are incorrect
  - For example: squaring both sides
    - $x + 1 = 3$ has one solution $x = 2$
    - $(x + 1)^2 = 3^2$ has two solutions $x = 2$ and $x = -4$
- Always **check your solutions** by substituting back into the **original equation**

How can I solve equations analytically where the unknown appears more than once?

- Sometimes it is possible to **simplify expressions** to make the **unknown appear only once**
- **Collect all terms** involving *x* on **one side** and try to simplify into one term
  - For **exponents** use
    - $a^{f(x)} \times a^{g(x)} = a^{f(x)+g(x)}$
    - $\frac{a^{f(x)}}{a^{g(x)}} = a^{f(x)-g(x)}$
    - $\left(a^{f(x)}\right)^{g(x)} = a^{f(x) \times g(x)}$
    - $a^{f(x)} = e^{f(x)\ln a}$
  - For **logarithms** use
    - $\log_a f(x) + \log_a g(x) = \log_a \left(f(x) \times g(x)\right)$
    - $\log_a f(x) - \log_a g(x) = \log_a \left(\frac{f(x)}{g(x)}\right)$
    - $n\log_a f(x) = \log_a \left(f(x)\right)^n$

How can I solve equations analytically when the equation can't be simplified?

- Sometimes it is **not possible to simplify** equations
- Most of these equations **cannot be solved analytically**
- A **special case** that can be solved is where the equation can be **transformed into a quadratic** using a substitution
  - These will have **three terms** and involve the same type of function
- **Identify** the **suitable substitution** by considering which **function is a square of another**
  - For example: the following can be transformed into $2y^2 + 3y - 4 = 0$
    - $2x^4 + 3x^2 - 4 = 0$ using $y = x^2$
    - $2x + 3\sqrt{x} - 4 = 0$ using $y = \sqrt{x}$
    - $\frac{2}{x^6} + \frac{3}{x^3} - 4 = 0$ using $y = \frac{1}{x^3}$
    - $2e^{2x} + 3e^x - 4 = 0$ using $y = e^x$
    - $2 \times 25^x + 3 \times 5^x - 4 = 0$ using $y = 5^x$
    - $2^{2x+1} + 3 \times 2^x - 4 = 0$ using $y = 2^x$
    - $2(x^3 - 1)^2 + 3(x^3 - 1) - 4 = 0$ using $y = x^3 - 1$
- To **solve**:
  - Make the **substitution** $y = f(x)$
  - **Solve** the **quadratic equation** $ay^2 + by + c = 0$ to get y_1 & y_2
  - **Solve** $f(x) = y_1$ and $f(x) = y_2$
    - Note that some equations might have **zero or several solutions**

Can I divide both sides of an equation by an expression?

- When dividing by an expression you must consider whether the **expression could be zero**
- Dividing by an expression that could be zero could result in you **losing solutions to the original equation**
  - For example: $(x+1)(2x-1)=3(x+1)$
    - If you divide both sides by $(x+1)$ you get $2x-1=3$ which gives $x=2$
    - However $x=-1$ is also a solution to the original equation
- To ensure you **do not lose solutions** you can:
  - **Split the equation into two equations**
    - One where the dividing expression equals zero: $x+1=0$
    - One where the equation has been divided by the expression: $2x-1=3$
  - **Make the equation equal zero and factorise**
    - $(x+1)(2x-1)-3(x+1)=0$
    - $(x+1)(2x-1-3)=0$ which gives $(x+1)(2x-4)=0$
    - Set each factor equal to zero and solve: $x+1=0$ and $2x-4=0$

Worked example

Find the exact solutions for the following equations:

a) $5-2\log_4 x=0$.

Rearrange using inverse functions

$5-2\log_4 x=0$

$2\log_4 x=5$ $\quad y=x-k \Leftrightarrow x=y+k$

$\log_4 x=\frac{5}{2}$ $\quad y=kx \Leftrightarrow x=\frac{y}{k}$

$x=4^{5/2}$ $\quad y=\log_a x \Leftrightarrow x=a^y$

$x=(\sqrt{4})^5$ $\quad a^{\frac{m}{n}}=(\sqrt[n]{a})^m$

$\boxed{x=32}$

b) $x=\sqrt{x+2}$.

Square both sides (Many-to-one function)

$x^2=x+2 \Rightarrow x^2-x-2=0$

$(x-2)(x+1)=0 \Rightarrow x=2$ or $x=-1$

Check whether each solution is valid

$x=2$: LHS $=2$ RHS $=\sqrt{2+2}=2$ ✓

$x=-1$: LHS $=-1$ RHS $=\sqrt{-1+2}=1$ ✗

$\boxed{x=2}$

c) $e^{2x}-4e^x-5=0$.

Notice $e^{2x} = (e^x)^2$, let $y = e^x$

$y^2 - 4y - 5 = 0 \Rightarrow (y+1)(y-5) = 0$

$y = -1$ or $y = 5$

Solve using $y = e^x$

$e^x = -1$ has no solutions as $e^x > 0$

$e^x = 5 \quad \therefore \quad x = \ln 5$

$\boxed{x = \ln 5}$

Solving Equations Graphically

How can I solve equations graphically?

- To solve $f(x) = g(x)$
  - One method is to **draw the graphs** $y = f(x)$ and $y = g(x)$
    - The **solutions** are the ***x*-coordinates** of the points of **intersection**
  - Another method is to **draw the graph** $y = f(x) - g(x)$ or $y = g(x) - f(x)$
    - The **solutions** are the **roots (zeros)** of this graph
      - This method is sometimes quicker as it involves **drawing only one graph**

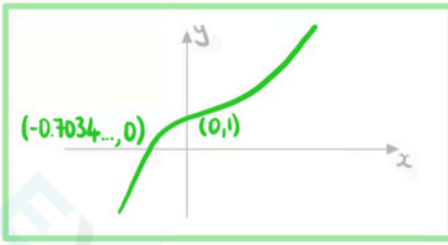
Why do I need to solve equations graphically?

- Some equations **cannot be solved analytically**
  - **Polynomials** of degree higher than 4
    - $x^5 - x + 1 = 0$
  - Equations involving **different types of functions**
    - $e^x = x^2$

Worked example

a) Sketch the graph $y = e^x - x^2$.

Sketch using GDC



b) Hence find the solution to $e^x = x^2$.

$e^x = x^2$ when $e^x - x^2 = 0$

Solution is the x-intercept of $y = e^x - x^2$

$\boxed{x = -0.703 \text{ (3sf)}}$

2.4.4 Modelling with Functions

Modelling with Functions

What is a mathematical model?

- A **mathematical model** simplifies a real-world situation so it can be described using mathematics
  - The model can then be used to make predictions
- **Assumptions** about the situation are made in order to simplify the mathematics
- Models can be **refined** (improved) if further information is available or if the model is compared to real-world data

How do I set up the model?

- The question could:
  - give you the equation of the model
  - tell you about the relationship
    - It might say the relationship is linear, quadratic, etc
  - ask you to suggest a **suitable model**
    - Use your knowledge of each model
    - E.g. if it is compound interest then an exponential model is the most appropriate
- You may have to determine a **reasonable domain**
  - Consider real-life context
    - E.g. if dealing with hours in a day then $0 \leq t < 24$
    - E.g. if dealing with physical quantities (such as length) then, $x > 0$
  - Consider the **possible ranges**
    - If the outcome cannot be negative then you want to choose a domain which corresponds to a range with no negative values
    - **Sketching the graph** is helpful to determine a suitable domain

Which models might I need to use?

- You could be given any model and be expected to use it
- Common models include:
  - **Linear**
    - Arithmetic sequences
    - Linear regression
  - **Quadratic**
    - Projectile motion
    - The height of a cable supporting a bridge
    - Profit
  - **Exponential**
    - Geometric sequences
    - Exponential growth and decay
    - Compound interest

- **Logarithmic**
  - Richter scale for the magnitude of earthquakes
- **Rational**
  - Temperature of a cup of coffee
- **Trigonometric**
  - The depth of a tide

How do I use a model?

- You can use a model by substituting in values for the variable to **estimate outputs**
  - For example: Let $h(t)$ be the height of a football t seconds after being kicked
    - $h(3)$ will be an estimate for the height of the ball 3 seconds after being kicked
- Given an **output** you can **form an equation** with the model to **estimate the input**
  - For example: Let $P(n)$ be the profit made by selling n items
    - Solving $P(n) = 100$ will give you an estimate for the number of items needing to be sold to make a profit of 100
- If your variable is **time** then substituting $t = 0$ will give you the **initial value** according to the model
- Fully understand the **units for the variables**
  - If the units of P are measured in **thousand dollars** then $P = 3$ represents $3000
- Look out for **key words** such as:
  - Initially
  - Minimum/maximum
  - Limiting value

What do I do if some of the parameters are unknown?

- A general method is to **form equations** by substituting in given values
  - You can form **multiple equations** and **solve them simultaneously** using your GDC
  - This method **works for all models**
- The **initial value** is the value of the function when the variable is 0
  - This is **normally one of the parameters** in the equation of the model

Worked example

The temperature, T°C, of a cup of coffee is monitored. Initially the temperature is 80°C and 5 minutes later it is 40°C . It is suggested that the temperature follows the model:

$$T(t) = A\mathrm{e}^{kt} + 16,\ t \geq 0.$$

where t is the time, in minutes, after the coffee has been made.

a) State the value of A.

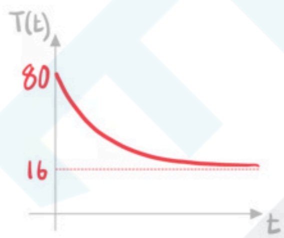
Initially temperature is 80°C

$T(0) = 80$

$Ae^{-k(0)} + 16 = 80$

$A + 16 = 80$

$\boxed{A = 64}$



b) Find the exact value of k.

$t = 5,\ T = 40$

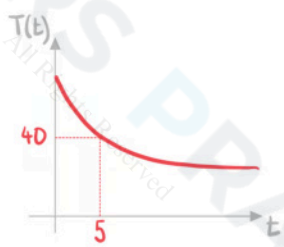
$40 = 64e^{5k} + 16$

$64e^{5k} = 24$

$e^{5k} = \frac{3}{8}$

$5k = \ln \frac{3}{8}$

$\boxed{k = \frac{1}{5} \ln \frac{3}{8}}$



c) Find the time taken for the temperature of the coffee to reach 30°C.

Find t such that $T(t) = 30$

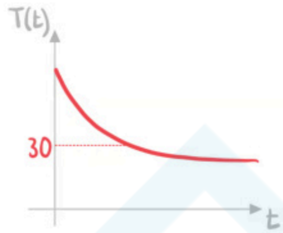
$30 = 64e^{kt} + 16$ Leave as k until the end to save writing $\frac{1}{5}\ln\frac{3}{8}$ each time

$64e^{kt} = 14$

$e^{kt} = \frac{7}{32}$

$kt = \ln\frac{7}{32}$

$t = \frac{\ln\frac{7}{32}}{k} = \frac{\ln\frac{7}{32}}{\frac{1}{5}\ln\frac{3}{8}} = 7.7476...$



$\boxed{7.75 \text{ minutes (3sf)}}$