

DP IB Maths: AI HL

2.4 Functions Toolkit

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★ 2.4.1 Composite & Inverse Functions



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Composite Functions

What is a composite function?

- A composite function is where a function is applied to another function
- A composite function can be denoted
 - $(f \circ g)(x)$
 - fg(x)
 - f(g(x))
- The order matters
 - $(f \circ g)(x)$ means:
 - First apply g to x to get g(x)
 - Then apply f to the previous output to get f(g(x))
 - Always start with the function closest to the variable
 - $(f \circ g)(x)$ is not usually equal to $(g \circ f)(x)$

How do I find the domain and range of a composite function?

- The domain of $f \circ g$ is the set of values of x...
 - which are a subset of the domain of g
 - which maps g to a value that is in the **domain of f**
- The range of $f \circ g$ is the set of values of $x \dots$
 - which are a subset of the range of f
 - found by applying f to the range of g
- To find the **domain** and **range** of $f \circ g$
 - First find the **range of g**
 - Restrict these values to the values that are within the domain of f
 - The **domain** is the set of values that **produce the restricted range** of g
 - The **range** is the set of values that are **produced using the restricted range** of g as the domain for f
- For example: let f(x) = 2x + 1, $-5 \le x \le 5$ and $g(x) = \sqrt{x}$, $1 \le x \le 49$
 - The range of g is $1 \le g(x) \le 7$
 - **Restricting** this to fit the **domain of** *f* results in $1 \le g(x) \le 5$
 - The domain of $f \circ g$ is therefore $1 \le x \le 25$
 - These are the values of x which map to $1 \le g(x) \le 5$
 - The range of $f \circ g$ is therefore $3 \le (f \circ g)(x) \le 11$
 - These are the values which f maps $1 \le g(x) \le 5$ to



Worked example

Given $f(x) = \sqrt{x+4}$ and g(x) = 3+2x: a) Write down the value of $(g \circ f)(12)$.

> First apply function closest to input $(g \circ f)(12) = g(f(12))$ $f(12) = \sqrt{12+4} = \sqrt{16} = 4$ g(4) = 3 + 2(4) = 11 $(g \circ f)(12) = 11$

b) Write down an expression for $(f \circ g)(x)$.

First apply function closest to input $(f \circ g)(x) = f(g(x))$ = f(3+2x) $= \sqrt{3+2x+4}$ $(f \circ g)(x) = \sqrt{7+2x}$

c) Write down an expression for $(g \circ g)(x)$.

$$(g \circ g)(x) = g(g(x))$$

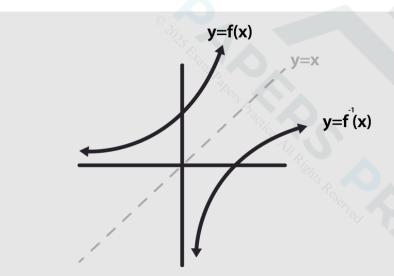
= $g(3 + 2x)$
= $3 + 2(3 + 2x)$
= $3 + 6 + 4x$
($g \circ g$)(x) = $9 + 4x$



Inverse Functions

What is an inverse function?

- Only one-to-one functions have inverses
- A function has an inverse if its graph passes the **horizontal line test**
 - Any horizontal line will intersect with the graph at most once
- The identity function id maps each value to itself
 - $\operatorname{id}(x) = x$
- If $f \circ g$ and $g \circ f$ have the same effect as the identity function then f and g are inverses
- Given a function f(x) we denote the inverse function as $f^{-1}(x)$
- An inverse function **reverses the effect** of a function
 - f(2) = 5 means $f^{-1}(5) = 2$
- Inverse functions are used to solve equations
 - The solution of f(x) = 5 is $x = f^{-1}(5)$
- A composite function made of f and f^{-1} has the same effect as the identity function
 - $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$



What are the connections between a function and its inverse function?

- The domain of a function becomes the range of its inverse
- The range of a function becomes the domain of its inverse
- The graph of $y = f^{-1}(x)$ is a **reflection** of the graph y = f(x) in the line y = x
 - Therefore solutions to f(x) = x or $f^{-1}(x) = x$ will also be solutions to $f(x) = f^{-1}(x)$
 - There could be other solutions to $f(x) = f^{-1}(x)$ that don't lie on the line y = x



How do I find the inverse of a function?

• STEP 1: Swap the x and y in y = f(x)

• If
$$y = f^{-1}(x)$$
 then $x = f(y)$

- STEP 2: Rearrange X = f(y) to make *Y* the subject
- Note this can be done in any order
 - Rearrange y = f(x) to make x the subject
 - Swap X and Y

Can many-to-one functions ever have inverses?

- You can restrict the domain of a many-to-one function so that it has an inverse
 - Choose a subset of the domain where the function is one-to-one
 - The inverse will be determined by the restricted domain
 - Note that a many-to-one function can only have an inverse if its domain is restricted first
- For quadratics use the vertex as the upper or lower bound for the restricted domain
 - For $f(x) = x^2$ restrict the domain so 0 is either the maximum or minimum value
 - For example: $X \ge 0$ or $X \le 0$
 - For f(x) = a(x h)² + k restrict the domain so h is either the maximum or minimum value
 For example: x > h or x < h
- For trigonometric functions use part of a cycle as the restricted domain
 - For $f(x) = \sin x$ restrict the domain to half a cycle between a maximum and a minimum
 - For example: $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$
 - For $f(x) = \cos x$ restrict the domain to half a cycle between maximum and a minimum
 - For example: $0 \le x \le \pi$
 - For $f(x) = \tan x$ restrict the domain to one cycle between two asymptotes
 - For example: $-\frac{\pi}{2} < x < \frac{\pi}{2}$

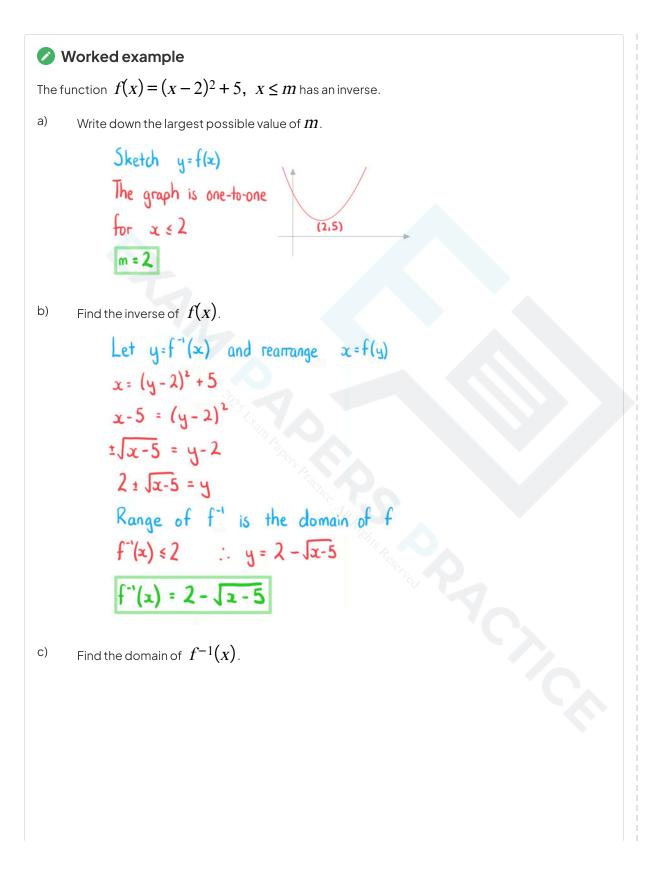
How do I find the inverse function after restricting the domain?

- The range of the inverse is the same as the restricted domain of the original function
- The inverse function is determined by the restricted domain
- Restricting the domain differently will change the inverse function
- Use the range of the inverse to help find the inverse function
 - Restricting the domain of $f(x) = x^2$ to $x \ge 0$ means the range of the inverse is $f^{-1}(x) \ge 0$

Therefore
$$f^{-1}(x) = \sqrt{x}$$

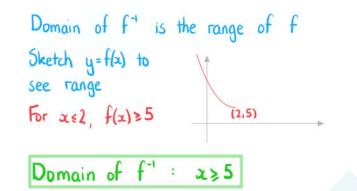
- Restricting the domain of $f(x) = x^2$ to $x \le 0$ means the range of the inverse is $f^{-1}(x) \le 0$
 - Therefore $f^{-1}(x) = -\sqrt{x}$





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d) Find the value of k such that f(k) = 9.

