DP IB Maths: AI HL

2.3 Modelling with Functions

Contents

2.3.1 Linear Models

Linear Models

What are the parameters of a linear model?

- A **linear model** is of the form $f(x) = mx + c$
- The *m* represents the **rate of change** of the function
  - This is the amount the function increases/decreases when *x* increases by 1
    - If the function is increasing *m* is positive
    - If the function is decreasing *m* is negative
  - When the model is represented as a graph this is the **gradient** of the line
- The c represents the value of the function when $x = 0$
  - This is the value of the function when the independent variable is not present
  - This is usually referred to as the initial value
  - When the model is represented as a graph this is the **y-intercept** of the line

What can be modelled as a linear model?

- If the graph of the data resembles a **straight line**
- Anything with a **constant** rate of change
  - $C(d)$ is the taxi charge for a journey of *d* km
  - $B(m)$ is the monthly mobile phone bill when *m* minutes have been used
  - $R(d)$ is the rental fee for a car used for *d* days
  - $d(t)$ is the distance travelled by a car moving at a constant speed for *t* seconds

What are possible limitations of a linear model?

- Linear models continuously increase (or decrease) at the same rate
  - In real-life this might not be the case
  - The function might reach a maximum (or minimum)
- If the value of *m* is negative then for some inputs the function will predict negative values
  - In some real-life situations negative values will not make sense
  - To overcome this you can decide on an appropriate domain so that the outputs are never negative

Worked example

The total cost, C, in New Zealand dollars (NZD), of a premium gym membership at FitFirst can be modelled by the function

$$C = 14.95t + 30, \quad t \geq 0$$

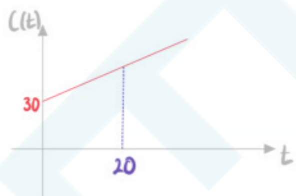
where t is the time in weeks.

a) Calculate the cost of the gym membership for 20 weeks.

Substitute $t = 20$

$C(20) = 14.95(20) + 30$

$\boxed{329 \text{ NZD}}$



b) Find the number of weeks it takes for the total cost to exceed 1500 NZD.

Substitute $C = 1500$

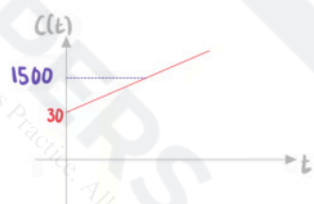
$1500 = 14.95t + 30$

$14.95t = 1470$

$t = \dfrac{1470}{14.95} = 98.327..$

Round up to the next integer

$\boxed{99 \text{ weeks}}$



c) Under new management, FitFirst changes the initial payment to 20 NZD and the weekly cost to 19.25 NZD. Write the new cost function after these changes have been.

$C(t) = mt + c$

m is the constant rate per week $m = 19.25$

c is the initial cost $c = 20$

$\boxed{C(t) = 19.25t + 20}$

2.3.2 Quadratic & Cubic Models

Quadratic Models

What are the parameters of a quadratic model?

- A quadratic model is of the form $f(x) = ax^2 + bx + c$
- The *c* represents the value of the function when $x = 0$
  - This is the value of the function when the independent variable is not present
  - This is usually referred to as the initial value
- The *a* has the biggest impact on the rate of change of the function
  - If *a* has a large absolute value then the rate of change varies rapidly
  - If *a* has a small absolute value then the rate of change varies slowly
- The maximum (or minimum) of the function occurs when $x = -\dfrac{b}{2a}$
  - This is given in the **formula booklet** as the **axis of symmetry**

What can be modelled as a quadratic model?

- If the graph of the data resembles a $\cup$ or $\cap$ shape
- These can be used if the graph has a single maximum or minimum
  - $H(t)$ is the vertical height of a football t seconds after being kicked
  - $A(x)$ is the area of rectangle of length x cm that can be made with a 20 cm length of string

What are possible limitations of a quadratic model?

- A quadratic has either a maximum or a minimum but **not both**
  - This means one end is **unbounded**
  - In real-life this might not be the case
  - The function might have both a maximum and a minimum
  - To overcome this you can decide on an appropriate domain so that the outputs are within a range
- Quadratic graphs are **symmetrical**
  - This might not be the case in real-life

Worked example

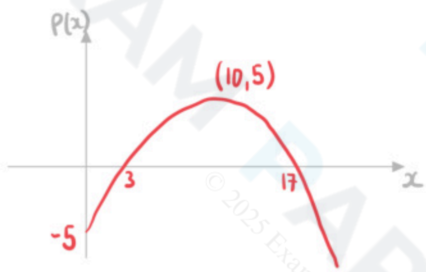
A company sells unicorn toys. The profit, $\pounds P$, made by selling one unicorn toy can be modelled by the function

$$P(x) = \frac{1}{10}(-x^2 + 20x - 50)$$

where x is the selling price of the toy.

Find the selling price which maximises profit. State the maximum profit.

Sketch on GDC and find the maximum point



$\boxed{\text{Selling price } \pounds 10 \quad \text{Maximum profit } \pounds 5}$

Cubic Models

What are the parameters of a cubic model?

- A **cubic model** is of the form $f(x) = ax^3 + bx^2 + cx + d$
- The *d* represents the value of the function when $x = 0$
  - This is the value of the function when the independent variable is not present
  - This is usually referred to as the initial value
- The *a* has the biggest impact on the rate of change of the function
  - If *a* has a large absolute value then the rate of change varies rapidly
  - If *a* has a small absolute value then the rate of change varies slowly

What can be modelled as a cubic model?

- If the graph of the data has exactly one maximum and one minimum within an interval
- If the graph is monotonic with no maximum or minimum
  - $D(t)$ is the vertical distance below starting point of a bungee jumper *t* seconds after jumping
  - $V(x)$ is the volume of a cuboid of length *x* cm that can be made with a 200 cm^2 of cardboard

What are possible limitations of a cubic model?

- Cubic graphs have **no global maximum or minimum**
  - This means the function is **unbounded**
  - In real-life this might not be the case
  - The function might have a maximum or minimum
  - To overcome this you can decide on an appropriate domain so that the outputs are within a range

Worked example

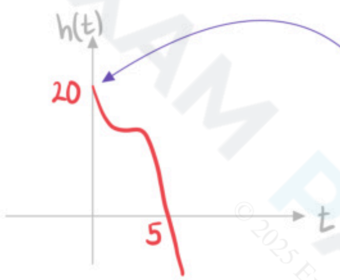
The vertical height of a child above the ground, h metres, as they go down a water slide can be modelled by the function

$$h(t) = \frac{4}{7}(35 - 12t + 6t^2 - t^3),$$

where t is the time in seconds after the child enters the slide.

a) State the vertical height of the slide.

Sketch on GDC

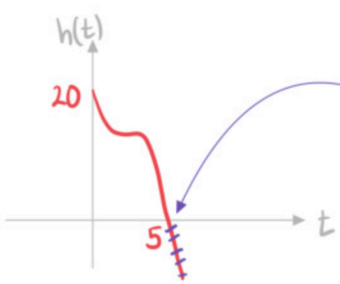


Height of slide is the initial height of the child

$\boxed{20 \text{ m}}$

b) Given that the child reaches the ground at the bottom of the slide, find the domain of the function.

Sketch on GDC



Child reaches the ground so function stops here

$\boxed{0 \leq t \leq 5}$

2.3.3 Exponential Models

Exponential Models

What are the parameters of an exponential model?

- An **exponential model** is of the form
  - $f(x) = ka^x + c$ or $f(x) = ka^{-x} + c$ for $a > 0$
  - $f(x) = ke^{rx} + c$
    - Where e is the mathematical constant 2.718…
  - The *c* represents the **boundary** for the function
    - It can never be this value
  - The *a* or *r* describes the **rate of growth or decay**
    - The bigger the value of *a* or the absolute value of *r* the faster the function increases/decreases

What can be modelled as an exponential model?

- Exponential growth or decay
  - Exponential **growth** is represented by
    - a^x where $a > 1$
    - a^{-x} where $0 < a < 1$
    - e^{rx} where $r > 0$
  - Exponential **decay** is represented by
    - a^x where $0 < a < 1$
    - a^{-x} where $a > 1$
    - e^{rx} where $r < 0$
- They can be used when there a **constant percentage increase or decrease**
  - Such as functions generated by **geometric sequences**
- Examples include:
  - *V*(*t*) is the value of car after *t* years
  - *S*(*t*) is the amount in a savings account after *t* years
  - *B*(*t*) is the amount of bacteria on a surface after *t* seconds
  - *T*(*t*) is the temperature of a kettle *t* minutes after being boiled

What are possible limitations of an exponential model?

- An exponential growth model does not have a maximum
  - In real-life this might not be the case
  - The function might reach a maximum and stay at this value
- Exponential models are **monotonic**
  - In real-life this might not be the case
  - The function might **fluctuate**

How can I find the half-life using an exponential model?

- You may need to find the **half-life** of a substance
  - This is the time taken for the mass of a substance to halve
- Given an exponential model $f(t) = ka^{-t}$ or $f(t) = k\mathrm{e}^{-rt}$ the half-life is the value of t such that:
  - $f(t) = \frac{k}{2}$
  - You can solve for t using your GDC
- For $f(t) = ka^{-t}$ the half-life is given by $t = \frac{\ln 2}{\ln a}$
  - $\frac{k}{2} = ka^{-t}$
  - $a^{t} = 2$
  - $t \ln a = \ln 2$
- For $f(t) = k\mathrm{e}^{-rt}$ the half-life is given by $t = \frac{\ln 2}{r}$
  - $\frac{k}{2} = k\mathrm{e}^{-rt}$
  - $\mathrm{e}^{rt} = 2$
  - $rt = \ln 2$

Worked example

The value of a car, V (NZD), can be modelled by the function

$$V(t) = 25125 \times 0.8^{t} + 8500, \quad t \geq 0$$

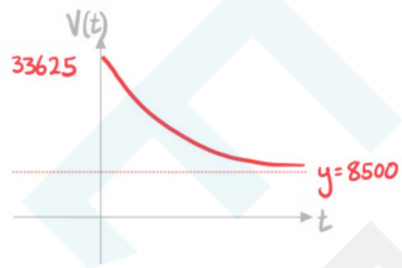
where t is the age of the car in years.

a) State the initial value of the car.

Initial value is when $t = 0$

$V(0) = 25125 \times 0.8^{0} + 8500$

$\boxed{33625 \text{ NZD}}$



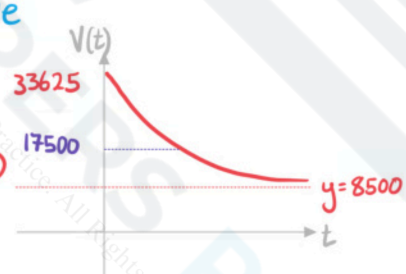
b) Find the age of the car when its value is 17500 NZD.

Set $V(t) = 17500$ and solve on GDC

$25125 \times 0.8^{t} + 8500 = 17500$

$t = 4.6007...$

$\boxed{4.60 \text{ years}}$



2.3.4 Direct & Inverse Variation

Direct Variation

What is direct variation?

- Two variables are said to **vary directly** if their **ratio is constant** (*k*)
  - This is also called **direct proportion**
- If y and x^n (for positive integer *n*) vary **directly** then:
  - It is denoted as $y \propto x^n$
  - $y = kx^n$ for some constant *k*
    - This can be written as $\frac{y}{x^n} = k$
- The graphs of these models always **start at the origin**

How do I solve direct variation problems?

- Identify which two variables vary directly
  - It might not be x and y
  - It could be x^3 and y
- Use the given information to find their **constant ratio *k***
  - Also called **constant of proportionality**
  - **Substitute** the given values of x and y into your formula
  - **Solve** to find *k*
- Write the equation which models their relationship
  - $y = kx^n$
- You can then use the equation to solve problems

Worked example

A computer program sorts a list of numbers into ascending order. The time it takes, t milliseconds, varies directly with the square of the number of items, n, in the list. The computer program takes 48 milliseconds to order a list with 8 items.

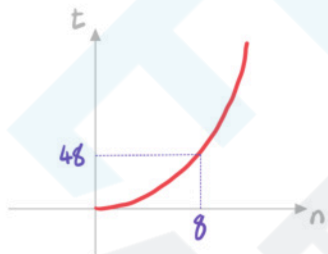
a) Find an equation connecting t and n.

Identify the variables that vary directly

$t \propto n^2$

Form an equation

$t = kn^2$



Use $t = 48$ and $n = 8$ to find the value of k

$48 = k(8)^2$

$64k = 48$

$k = \frac{48}{64} = 0.75$

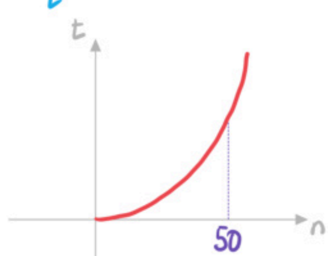
$\boxed{t = 0.75n^2}$

b) Find the time it takes to order a list of 50 numbers.

Substitute $n = 50$ into the equation

$t = 0.75(50)^2$

$\boxed{1875 \text{ milliseconds}}$



Inverse Variation

What is inverse variation?

- Two variables are said to **vary inversely** if their **product is constant (*k*)**
  - This is also called **inverse proportion**
- If y and x^n (for positive integer *n*) vary **inversely** then:
  - It is denoted $y \propto \frac{1}{x^n}$
  - $y = \frac{k}{x^n}$ for some constant *k*
    - This can be written $x^n y = k$
- The graphs of these models all have a **vertical asymptote** at the ***y*-axis**
  - This means that as x gets closer to 0 the absolute value of y gets further away from 0
  - x can never equal 0

How do I solve inverse variation problems?

- Identify which two variables vary inversely
  - It might not be x and y
  - It could be x^3 and y
- Use the given information to find their **constant product *k***
  - Also called **constant of proportionality**
  - **Substitute** the given values of x and y into your formula
  - **Solve** to find *k*
- Write the equation which models their relationship
  - $y = \frac{k}{x^n}$
- You can then use the equation to solve problems

Worked example

The time, t hours, it takes to complete a project varies inversely to the number of people working on it, n. If 4 people work on the project it takes 70 hours to complete.

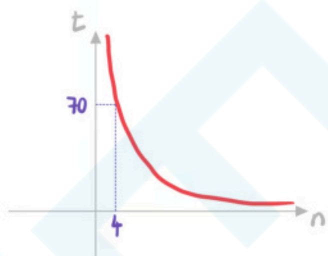
a) Write an equation connecting t and n.

Identify the variables that vary directly

$t \propto \frac{1}{n}$

Form an equation

$t = \frac{k}{n}$



Use $t = 70$ and $n = 4$ to

Find the value of k

$70 = \frac{k}{4}$

$k = 4 \times 70 = 280$

$\boxed{t = \frac{280}{n}}$

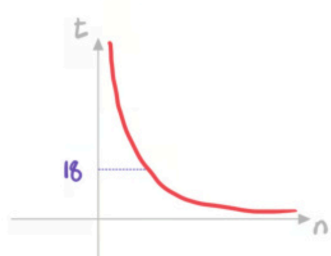
b) Given that the project needs to be completed within 18 hours, find the minimum number of people needed to work on it.

Substitute $t = 18$ into the equation

$18 = \frac{280}{n}$

$n = \frac{280}{18} = 15.55...$

$\boxed{\text{16 people}}$



2.3.5 Sinusoidal Models

Sinusoidal Models

What are the parameters of a sinusoidal model?

- A **sinusoidal model** is of the form
  - $f(x) = a\sin(b(x - c)) + d$
  - $f(x) = a\cos(b(x - c)) + d$
- The *a* represents the **amplitude** of the function
  - The bigger the value of *a* the bigger the **range** of values of the function
- The *b* determines the **period** of the function
  - The bigger the value of *b* the quicker the function repeats a cycle
  - The period is $\frac{360°}{b}$ (in degrees) or $\frac{2\pi}{b}$ (in radians)
- The *c* represents the **phase shift**
  - This is a horizontal translation by *c* units
- The *d* represents the **principal axis**
  - This is the line that the function fluctuates around

What can be modelled as a sinusoidal model?

- Anything that oscillates (fluctuates periodically)
- Examples include:
  - *D*(*t*) is the depth of water at a shore *t* hours after midnight
  - *T*(*d*) is the temperature of a city *d* days after the 1st January
  - *H*(*t*) is vertical height above ground of a person *t* second after entering a Ferris wheel

What are possible limitations of a sinusoidal model?

- The amplitude is the same for each cycle
  - In real-life this might not be the case
  - The function might get closer to the principal axis over time
- The period is the same for each cycle
  - In real-life this might not be the case
  - The time to complete a cycle might change over time

Worked example

The water depth, D, in metres, at a port can be modelled by the function

$$D(t) = 3\sin\left(\frac{\pi}{12}(t-2)\right) + 12, \quad 0 \le t < 24$$

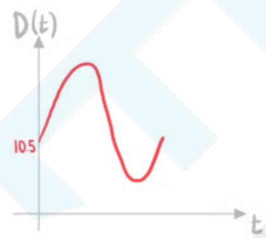
where t is the elapsed time, in hours, since midnight.

a) Write down the depth of the water at midnight.

Substitute $t = 0$ for midnight

$D(0) = 3\sin\left(\frac{\pi}{12}(t-2)\right) + 12$

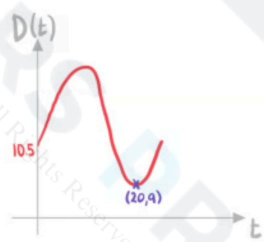
$\boxed{10.5 \text{ m}}$



b) Find the minimum water depth and the number of hours after midnight that this depth occurs.

Use GDC to find the minimum

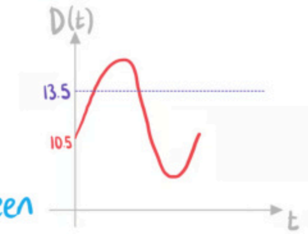
Minimum = 9 m
20 hours after midnight



c) Calculate how long the water depth is at least 13.5 metres each day.

Use GDC to find $D(t) = 13.5$

$3\sin\left(\frac{\pi}{12}(t-2)\right) + 12 = 13.5$



$t = 4$ and $t = 12$

Find the difference between the times

$12 - 4 = 8$

$\boxed{8 \text{ hours}}$

2.3.6 Strategy for Modelling Functions

Modelling with Functions

What is a mathematical model?

- A **mathematical model** simplifies a real-world situation so it can be described using mathematics
  - The model can then be used to make predictions
  - Be aware that **extrapolating** (making predictions outside of the range of the data) is not considered to be accurate
- **Assumptions** about the situation are made in order to simplify the mathematics
- Models can be **refined** (improved) if further information is available or if the model is compared to real-world data

How do I set up the model?

- The question could:
  - give you the equation of the model
  - tell you about the relationship
    - It might say the relationship is linear, quadratic, etc
  - ask you to suggest a **suitable model**
    - Use your knowledge of each model
    - E.g. if it is compound interest then an exponential model is the most appropriate
- You may have to determine a **reasonable domain**
  - Consider real-life context
    - E.g. if dealing with hours in a day then $0 \le t < 24$
    - E.g. if dealing with physical quantities (such as length) then $x > 0$
  - Consider the **possible ranges**
    - If the outcome cannot be negative then you want to choose a domain which corresponds to a range with no negative values
    - **Sketching the graph** is helpful to determine a suitable domain

Which models do I need to know?

- Linear
- Piecewise (linear & non-linear)
- Quadratic
- Cubic
- Exponential
- Natural logarithmic
- Logistic
- Direct variation
- Inverse variation
- Sinusoidal

Worked example

A cliff has a height h metres above the ground. A stone is projected from the edge of the cliff and it travels through the air until it hits the ground and stops. The vertical height, in metres, of the stone above the ground t seconds after being thrown is given by the function:

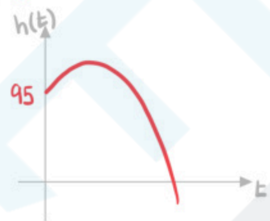
$$h(t) = 95 + 6t - 5t^2.$$

a) State the initial value of h.

Initial value is the height of the cliff

$h(0) = 95 + 6(0) - 5(0)^2$

$\boxed{95 \text{ m}}$



b) Determine the domain of $h(t)$.

Stone stops at ground when $h(t) = 0$

$95 + 6t - 5t^2 = 0$

$t = 5$ or $t = -3.8$

Reject as time can't be negative

$\boxed{0 \leq t \leq 5}$

Finding Parameters

What do I do if some of the parameters are unknown?

- For some models you can use your knowledge to find unknown parameters directly from the information given
  - For a **linear** model $f(x) = mx + c$
    - m is the rate of change, or gradient
    - c is the initial value
  - For a **quadratic** model, $f(x) = ax^2 + bx + c$
    - $x = \frac{-b}{2a}$ is the axis of symmetry (this is given in the formula booklet) and is the x-value of the minimum/ maximum point
    - c is the initial value
  - For a **cubic** model, $f(x) = ax^3 + bx^2 + cx + d$
    - d is the initial value
  - For an **exponential** model, $f(x) = ka^x + c$
    - $k + c$ is the initial value
    - $y = c$ is the horizontal asymptote, so c is a boundary of the model
  - For a **sinusoidal** model $f(x) = a\sin(bx) + d$
    - a is the amplitude
    - $y = d$ is the principal axis
    - $\frac{360}{b}$ is the period
- A general method is to form equations by substituting in given values
  - You can form multiple equations and solve them **simultaneously using your GDC**
    - You could be expected to solve a system of up to **three simultaneous equations** of three unknowns
  - This method works for all models
- The **initial value** is the value of the function when x (or the independent variable) is 0
  - This is often one of the parameters in the equation of the model

Worked example

The temperature, T °C, of a cup of coffee is monitored. Initially the temperature is 80°C and 5 minutes later it is 40°C. It is suggested that the temperature follows the model:

$$T(t) = ka^{-t} + 16, \quad t \geq 0$$

where t is the time, in minutes, after the coffee has been made.

a) State the value of k.

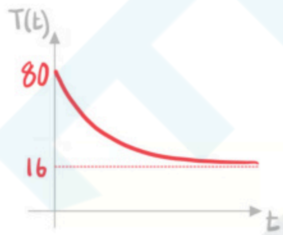
Initially temperature is 80°C

$T(0) = 80$

$ka^{-0} + 16 = 80$

$k + 16 = 80$

$\boxed{k = 64}$



b) Find the value of a.

After 5 minutes the temperature is 40°C

$T(5) = 40$

$64a^{-5} + 16 = 40$

Solve using GDC

$a = 1.21672...$

$\boxed{a = 1.22 \quad (3sf)}$