DP IB Maths: AA HL

2.3 Functions Toolkit

Contents

2.3.1 Language of Functions

Language of Functions

What is a mapping?

- A **mapping transforms** one set of values (**inputs**) into another set of values (**outputs**)
- Mappings can be:
  - **One-to-one**
    - Each input gets mapped to **exactly one unique** output
    - No two inputs are mapped to the same output
    - For example: A mapping that cubes the input
  - **Many-to-one**
    - Each input gets mapped to **exactly one** output
    - Multiple inputs can be mapped to the same output
    - For example: A mapping that squares the input
  - **One-to-many**
    - An input can be mapped to **more than one** output
    - No two inputs are mapped to the same output
    - For example: A mapping that gives the numbers which when squared equal the input
  - **Many-to-many**
    - An input can be mapped to **more than one** output
    - Multiple inputs can be mapped to the same output
    - For example: A mapping that gives the factors of the input

What is a function?

- A **function** is a mapping between two sets of numbers where **each input** gets mapped to **exactly one output**
  - The output does not need to be unique
- **One-to-one** and **many-to-one** mappings are functions
- A mapping is a function if its graph passes the **vertical line test**
  - Any **vertical line** will intersect with the graph **at most once**

What notation is used for functions?

- Functions are denoted using letters (such as f, v, g, etc)
  - A function is followed by a variable in a bracket
  - This shows the input for the function
  - The letter f is used most commonly for functions and will be used for the remainder of this revision note
- $f(x)$ represents an expression for the value of the function f when evaluated for the variable x
- Function notation gets rid of the need for words which makes it **universal**
  - $f = 5$ when $x = 2$ can simply be written as $f(2) = 5$

What are the domain and range of a function?

- The **domain** of a function is the set of values that are used as **inputs**
- A domain should be stated with a function
  - If a domain is not stated then it is assumed the domain is all the real values which would work as inputs for the function
  - Domains are expressed in terms of the input
    - $x \leq 2$
- The **range** of a function is the set of values that are given as **outputs**
  - The range depends on the domain
  - Ranges are expressed in terms of the output

- $f(x) \geq 0$

- To graph a function we use the **inputs as the *x*-coordinates** and the **outputs as the *y*-coordinates**
  - $f(2) = 5$ corresponds to the coordinates (2, 5)
- Graphing the function can help you visualise the range
- Common sets of numbers have special symbols:
  - $\mathbb{R}$ represents all the real numbers that can be placed on a number line
    - $x \in \mathbb{R}$ means x is a real number
  - $\mathbb{Q}$ represents all the rational numbers $\frac{a}{b}$ where *a* and *b* are integers and $b \neq 0$
  - $\mathbb{Z}$ represents all the integers (positive, negative and zero)
    - $\mathbb{Z}^+$ represents positive integers
  - $\mathbb{N}$ represents the natural numbers (0,1,2,3...)

What are piecewise functions?

- **Piecewise functions** are defined by different functions depending on which interval the input is in
  - E.g. $f(x) = \begin{cases} x+1 & x \leq 5 \\ 2x-4 & 5 < x < 10 \\ x^2 & 10 \leq x \leq 20 \end{cases}$
- The region for the individual functions **cannot overlap**
- To evaluate a piecewise function for a particular value $x = k$
  - Find which interval includes k
  - Substitute $x = k$ into the corresponding function
- The function **may or may not be continuous** at the ends of the intervals
  - In the example above the function is
    - continuous at $x = 5$ as $5 + 1 = 2(5) - 4$
    - not continuous at $x = 10$ as $2(10) - 4 \neq 10^2$

Worked example

For the function $f(x) = x^3 + 1,\ 2 \le x \le 10$:

a) write down the value of $f(7)$.

Substitute $x = 7$

$f(7) = 7^3 + 1$

$\boxed{f(7) = 344}$

b) find the range of $f(x)$.

Find the values of $x^3 + 1$ when $2 \le x \le 10$

$2 \le x \le 10$

$8 \le x^3 \le 1000$

$9 \le x^3 + 1 \le 1001$

$\boxed{9 \le f(x) \le 1001}$

2.3.2 Composite & Inverse Functions

Composite Functions

What is a composite function?

- A **composite function** is where a function is applied to another function
- A composite function can be denoted
  - $(f \circ g)(x)$
  - $fg(x)$
  - $f(g(x))$
- The order matters
  - $(f \circ g)(x)$ means:
    - First apply *g* to *x* to get $g(x)$
    - Then apply *f* to the previous output to get $f(g(x))$
    - Always start with the function **closest to the variable**
  - $(f \circ g)(x)$ is not usually equal to $(g \circ f)(x)$

How do I find the domain and range of a composite function?

- The domain of $f \circ g$ is the set of values of x...
  - which are a **subset** of the **domain of g**
  - which maps *g* to a value that is in the **domain of f**
- The range of $f \circ g$ is the set of values of x...
  - which are a **subset** of the **range of f**
  - found by **applying f** to the **range of g**
- To find the **domain** and **range** of $f \circ g$
  - First find the **range of g**
  - **Restrict** these values to the values that are **within the domain** of **f**
    - The **domain** is the set of values that **produce the restricted range** of *g*
    - The **range** is the set of values that are **produced using the restricted range** of *g* as the domain for *f*
- For example: let $f(x) = 2x + 1,\ -5 \le x \le 5$ and $g(x) = \sqrt{x},\ 1 \le x \le 49$
  - The **range of g** is $1 \le g(x) \le 7$
    - **Restricting** this to fit the **domain of f** results in $1 \le g(x) \le 5$
  - The **domain** of $f \circ g$ is therefore $1 \le x \le 25$
    - These are the values of *x* which map to $1 \le g(x) \le 5$
  - The **range** of $f \circ g$ is therefore $3 \le (f \circ g)(x) \le 11$
    - These are the values which *f* maps $1 \le g(x) \le 5$ to

Worked example

Given $f(x) = \sqrt{x+4}$ and $g(x) = 3 + 2x$:

a) Write down the value of $(g \circ f)(12)$.

First apply function closest to input

$(g \circ f)(12) = g(f(12))$

$f(12) = \sqrt{12+4} = \sqrt{16} = 4$

$g(4) = 3 + 2(4) = 11$

$\boxed{(g \circ f)(12) = 11}$

b) Write down an expression for $(f \circ g)(x)$.

First apply function closest to input

$(f \circ g)(x) = f(g(x))$

$= f(3 + 2x)$

$= \sqrt{3 + 2x + 4}$

$\boxed{(f \circ g)(x) = \sqrt{7 + 2x}}$

c) Write down an expression for $(g \circ g)(x)$.

$(g \circ g)(x) = g(g(x))$

$= g(3 + 2x)$

$= 3 + 2(3 + 2x)$

$= 3 + 6 + 4x$

$\boxed{(g \circ g)(x) = 9 + 4x}$

Inverse Functions

What is an inverse function?

- **Only one-to-one** functions have inverses
- A function has an inverse if its graph passes the **horizontal line test**
  - Any **horizontal line** will intersect with the graph **at most once**
- The **identity function** id maps each value to itself
  - $\mathrm{id}(x) = x$
- If $f \circ g$ and $g \circ f$ have the **same effect as the identity function** then f and g are **inverses**
- Given a function $f(x)$ we denote the **inverse function** as $f^{-1}(x)$
- An inverse function **reverses the effect** of a function
  - $f(2) = 5$ means $f^{-1}(5) = 2$
- Inverse functions are used to solve equations
  - The solution of $f(x) = 5$ is $x = f^{-1}(5)$
- A composite function made of f and f^{-1} has the **same effect as the identity function**
  - $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$

What are the connections between a function and its inverse function?

- The **domain of a function** becomes the **range of its inverse**
- The **range of a function** becomes the **domain of its inverse**
- The graph of $y = f^{-1}(x)$ is a **reflection** of the graph $y = f(x)$ in the line $y = x$
  - Therefore solutions to $f(x) = x$ or $f^{-1}(x) = x$ will also be solutions to $f(x) = f^{-1}(x)$
    - There could be other solutions to $f(x) = f^{-1}(x)$ that don't lie on the line $y = x$

How do I find the inverse of a function?

- STEP 1: **Swap** the x and y in $y = f(x)$
  - If $y = f^{-1}(x)$ then $x = f(y)$
- STEP 2: **Rearrange** $x = f(y)$ to make y the subject
- Note this can be done in any order
  - Rearrange $y = f(x)$ to make x the subject
  - Swap x and y

Can many-to-one functions ever have inverses?

- You can **restrict the domain** of a many-to-one function so that it has an inverse
- Choose a subset of the domain where the function is one-to-one
  - The inverse will be determined by the restricted domain
  - Note that a many-to-one function can **only** have an inverse if its domain is restricted first
- For **quadratics** – use the **vertex** as the upper or lower bound for the **restricted domain**
  - For $f(x) = x^2$ restrict the domain so 0 is either the maximum or minimum value
    - For example: $x \geq 0$ or $x \leq 0$
  - For $f(x) = a(x-h)^2 + k$ restrict the domain so h is either the maximum or minimum value
    - For example: $x \geq h$ or $x \leq h$
- For **trigonometric functions** – use part of a cycle as the **restricted domain**
  - For $f(x) = \sin x$ restrict the domain to half a cycle between a maximum and a minimum
    - For example: $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
  - For $f(x) = \cos x$ restrict the domain to half a cycle between maximum and a minimum
    - For example: $0 \leq x \leq \pi$
  - For $f(x) = \tan x$ restrict the domain to one cycle between two asymptotes
    - For example: $-\frac{\pi}{2} < x < \frac{\pi}{2}$

How do I find the inverse function after restricting the domain?

- The range of the inverse is the same as the restricted domain of the original function
- The inverse function is determined by the restricted domain
  - Restricting the domain differently will change the inverse function
- Use the range of the inverse to help find the inverse function
  - Restricting the domain of $f(x) = x^2$ to $x \geq 0$ means the range of the inverse is $f^{-1}(x) \geq 0$
    - Therefore $f^{-1}(x) = \sqrt{x}$
  - Restricting the domain of $f(x) = x^2$ to $x \leq 0$ means the range of the inverse is $f^{-1}(x) \leq 0$
    - Therefore $f^{-1}(x) = -\sqrt{x}$

Worked example

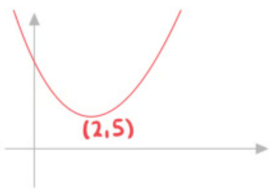
The function $f(x) = (x-2)^2 + 5, \;\; x \le m$ has an inverse.

a) Write down the largest possible value of m.

Sketch $y = f(x)$

The graph is one-to-one for $x \le 2$

$\boxed{m = 2}$



b) Find the inverse of $f(x)$.

Let $y = f^{-1}(x)$ and rearrange $x = f(y)$

$x = (y-2)^2 + 5$

$x - 5 = (y-2)^2$

$\pm\sqrt{x-5} = y - 2$

$2 \pm \sqrt{x-5} = y$

Range of f^{-1} is the domain of f

$f^{-1}(x) \le 2 \quad \therefore \; y = 2 - \sqrt{x-5}$

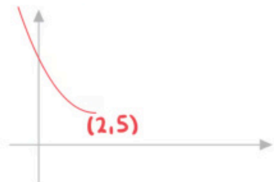
$\boxed{f^{-1}(x) = 2 - \sqrt{x-5}}$

c) Find the domain of $f^{-1}(x)$.

Domain of f^{-1} is the range of f

Sketch $y=f(x)$ to see range

For $x \leq 2$, $f(x) \geq 5$



$\boxed{\text{Domain of } f^{-1} : x \geq 5}$

d) Find the value of k such that $f(k)=9$.

Use inverse $f(a)=b \iff a=f^{-1}(b)$

$k=f^{-1}(9) = 2-\sqrt{9-5}$

$\boxed{k=0}$

2.3.3 Symmetry of Functions

Odd & Even Functions

What are odd functions?

- A function $f(x)$ is called **odd** if
  - $f(-x) = -f(x)$ for all values of x
- Examples of odd functions include:
  - Power functions with **odd powers**: x^{2n+1} where $n \in \mathbb{Z}$
    - For example: $(-x)^3 = -x^3$
  - Some **trig functions**: $\sin x$, $\operatorname{cosec} x$, $\tan x$, $\cot x$
    - For example: $\sin(-x) = -\sin x$
  - **Linear combinations** of odd functions
    - For example: $f(x) = 3x^5 - 4\sin x + \dfrac{6}{x}$

What are even functions?

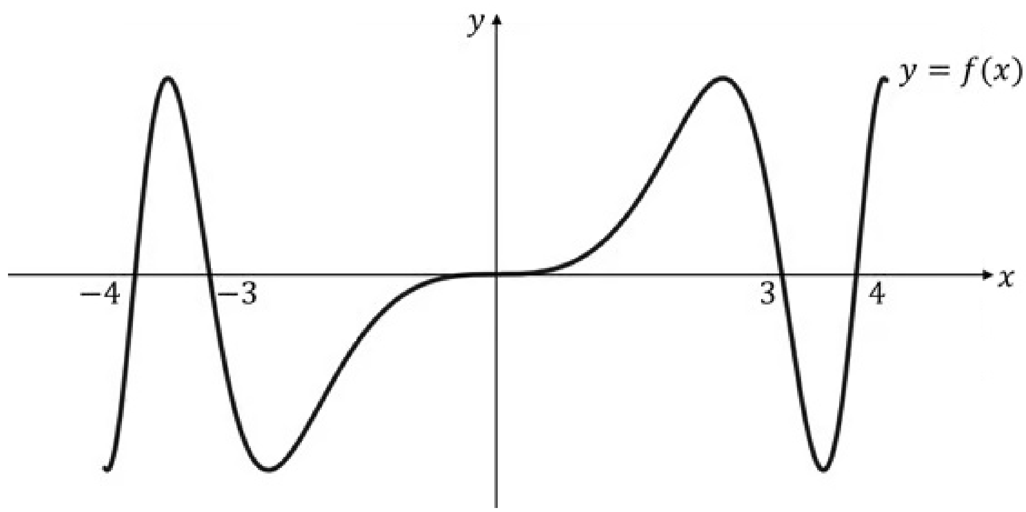
- A function $f(x)$ is called **even** if
  - $f(-x) = f(x)$ for all values of x
- Examples of even functions include:
  - Power functions with **even powers**: x^{2n} where $n \in \mathbb{Z}$
    - For example: $(-x)^4 = x^4$
  - Some **trig functions**: $\cos x$, $\sec x$
    - For example: $\cos(-x) = \cos x$
  - **Modulus function**: $|x|$
  - **Linear combinations** of even functions
    - For example: $f(x) = 7x^6 + 3|x| - 8\cos x$

What are the symmetries of graphs of odd & even functions?

- The graph of an **odd** function has **rotational symmetry**
  - The graph is unchanged by a **180° rotation** about the origin
- The graph of an **even** function has **reflective symmetry**
  - The graph is unchanged by a **reflection** in the ***y*-axis**

Worked example

a) The graph $y = f(x)$ is shown below. State, with a reason, whether the function f is odd, even or neither.



f is an odd function as its graph has rotational symmetry – it is unchanged by a 180° rotation about the origin.

b) Use algebra to show that $g(x) = x^3\sin(x) + 5\cos(x^5)$ is an even function.

g is even if $g(-x) = g(x)$ for all x

$$\begin{aligned} g(-x) &= (-x)^3\sin(-x) + 5\cos((-x)^5) \\ &= (-x^3)(-\sin(x)) + 5\cos(-x^5) \\ &= x^3\sin(x) + 5\cos(x^5) \\ &= g(x) \end{aligned}$$

x^3, x^5, $\sin x$ are odd

$\cos x$ is even

g is even as $g(-x) = g(x)$ for all x

Periodic Functions

What are periodic functions?

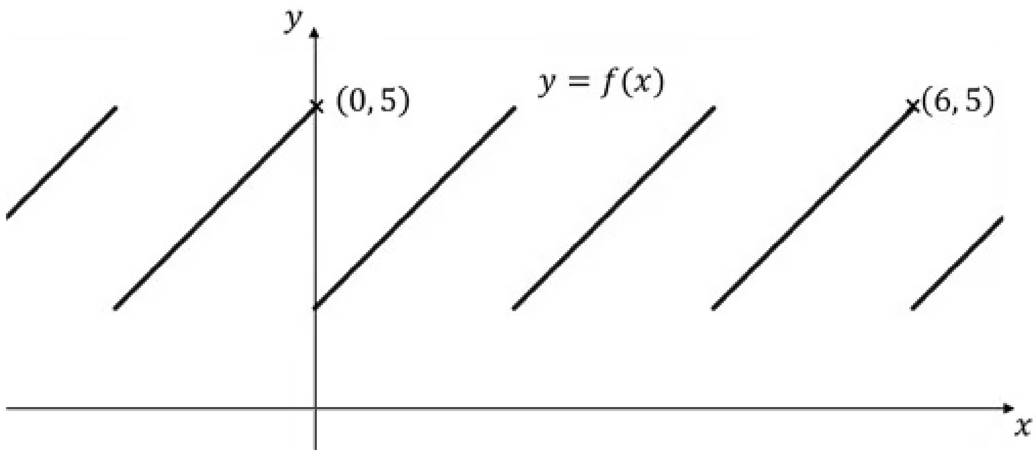
- A function $f(x)$ is called **periodic**, with **period *k***, if
  - $f(x+k)=f(x)$ for all values of x
- Examples of periodic functions include:
  - sin *x* & cos *x*: The period is 2π or 360°
  - tan *x*: The period is π or 180°
  - **Linear combinations** of periodic functions with the **same period**
    - For example: $f(x)=2\sin(3x)-5\cos(3x+2)$

What are the symmetries of graphs of periodic functions?

- The graph of a **periodic** function has **translational symmetry**
  - The graph is unchanged by **translations** that are **integer multiples of** $\begin{pmatrix} k \\ 0 \end{pmatrix}$
  - The means that the graph appears to **repeat** the same section (cycle) infinitely

Worked example

The graph $y = f(x)$ is shown below. Given that f is periodic, write down the period.



Period is the length of the interval of a single cycle

Between $x = 0$ and $x = 6$ there are 3 cycles

Period $= \frac{6-0}{3}$

$\boxed{\text{Period} = 2}$

Self-Inverse Functions

What are self-inverse functions?

- A function $f(x)$ is called **self-inverse** if
  - $(f \circ f)(x) = x$ for all values of x
  - $f^{-1}(x) = f(x)$
- Examples of self-inverse functions include:
  - **Identity function**: $f(x) = x$
  - **Reciprocal function**: $f(x) = \frac{1}{x}$
  - **Linear functions** with a **gradient of -1**: $f(x) = -x + c$

What are the symmetries of graphs of self-inverse functions?

- The graph of a **self-inverse** function has **reflective symmetry**
  - The graph is unchanged by a **reflection** in the line ***y* = *x***

Worked example

Use algebra to show the function defined by $f(x) = \dfrac{7x-5}{x-7}$, $x \neq 7$ is self-inverse.

Method 1: $f^{-1}(x)$

Let $y = f^{-1}(x)$ so $x = f(y)$

$$x = \frac{7y-5}{y-7}$$

$$(y-7)x = 7y-5$$

$$xy - 7x = 7y - 5$$

$$xy - 7y = 7x - 5$$

Isolate y on one side

$$(x-7)y = 7x-5$$

$$y = \frac{7x-5}{x-7}$$

$$\boxed{f^{-1}(x) = \frac{7x-5}{x-7} = f(x) \quad \therefore f \text{ is self-inverse}}$$

Method 2: $(f \circ f)(x)$

$$(f \circ f)(x) = f(f(x))$$

$$f(f(x)) = \frac{7f(x)-5}{f(x)-7}$$

$$= \frac{7\left(\frac{7x-5}{x-7}\right)-5}{\frac{7x-5}{x-7}-7}$$

$\times \dfrac{x-7}{x-7}$

$$= \frac{7(7x-5)-5(x-7)}{7x-5-7(x-7)}$$

$$= \frac{49x - \cancel{35} - 5x + \cancel{35}}{\cancel{7x} - 5 - \cancel{7x} + 49}$$

$$= \frac{\cancel{44}x}{\cancel{44}}$$

$$\boxed{(f \circ f)(x) = x \quad \therefore f \text{ is self-inverse}}$$

2.3.4 Graphing Functions

Graphing Functions

How do I graph the function *y* = *f*(*x*)?

- A point (a, b) lies on the graph $y = f(x)$ if $f(a) = b$
- The **horizontal axis** is used for the **domain**
- The **vertical axis** is used for the **range**
- You will be able to graph some functions by hand
- For some functions you will need to use your GDC
- You might be asked to graph the **sum** or **difference** of two functions
  - Use your GDC to graph $y = f(x) + g(x)$ or $y = f(x) - g(x)$
  - Just type the functions into the graphing mode

What is the difference between "draw" and "sketch"?

- If asked to sketch you should:
  - Show the general shape
  - Label any key points such as the intersections with the axes
  - Label the axes
- If asked to draw you should:
  - Use a pencil and ruler
  - Draw to scale
  - Plot any points **accurately**
  - Join points with a straight line or smooth curve
  - Label any key points such as the intersections with the axes
  - Label the axes

How can my GDC help me sketch/draw a graph?

- You use your GDC to plot the graph
  - Check the scales on the graph to make sure you see the full shape
- Use your GDC to find any key points
- Use your GDC to check specific points to help you plot the graph

Key Features of Graphs

What are the key features of graphs?

- You should be familiar with the following key features and know how to use your GDC to find them
- Local minimums/maximums
  - These are points where the graph has a minimum/maximum for a small region
  - They are also called **turning points**
    - This is where the graph changes its direction between upwards and downwards directions
  - A graph can have multiple local minimums/maximums
  - A local minimum/maximum is not necessarily the minimum/maximum of the whole graph
    - This would be called the **global** minimum/maximum
  - For quadratic graphs the minimum/maximum is called the **vertex**
- Intercepts
  - y – intercepts are where the graph crosses the y-axis
    - At these points $x = 0$
  - x – intercepts are where the graph crosses the x-axis
    - At these points $y = 0$
    - These points are also called the **zeros of the function** or **roots of the equation**
- Symmetry
  - Some graphs have lines of symmetry
    - A quadratic will have a vertical line of symmetry
- Asymptotes
  - These are lines which the graph will get closer to but not cross
  - These can be horizontal or vertical
    - Exponential graphs have horizontal asymptotes
    - Graphs of variables which vary inversely can have vertical and horizontal asymptotes

Worked example

Two functions are defined by

$$f(x) = x^2 - 4x - 5 \text{ and } g(x) = 2 + \frac{1}{x+1}.$$

a) Draw the graph $y = f(x)$.

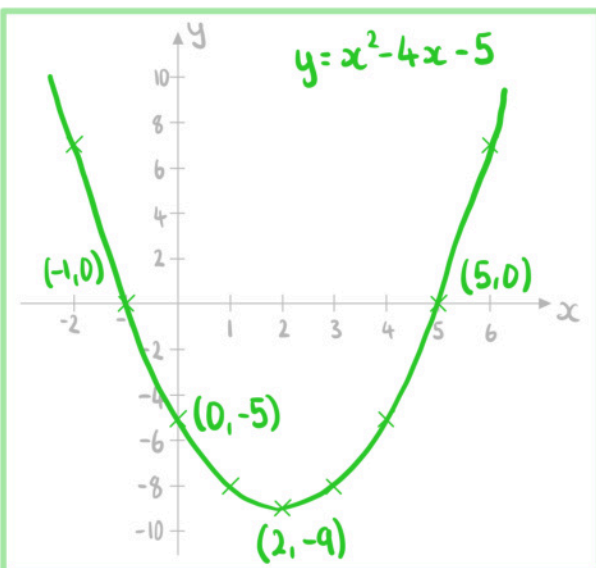
Draw means accurately

Use GDC to find vertex, roots and y-intercepts

Vertex = (2, -9)

Roots = (-1, 0) and (5, 0)

y-intercept = (0, -5)



b) Sketch the graph $y = g(x)$.

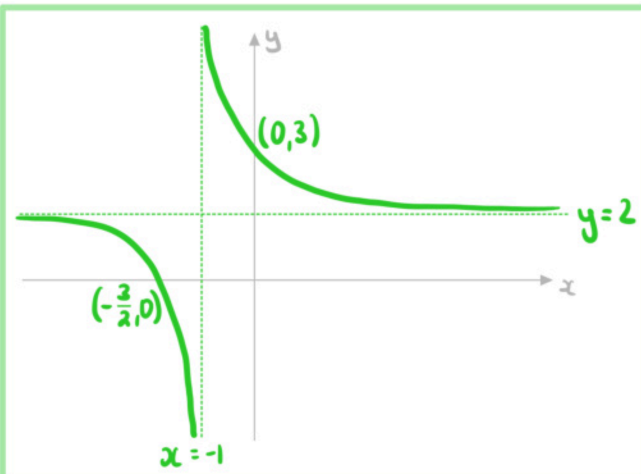
Sketch means rough but showing key points

Use GDC to find x and y-intercepts and asymptotes

x-intercept $= \left(-\frac{3}{2}, 0\right)$

y-intercept $= (0, 3)$

Asymptotes : $x = -1$ and $y = 2$



Intersecting Graphs

How do I find where two graphs intersect?

- Plot both graphs on your GDC
- Use the intersect function to find the intersections
- Check if there is more than one point of intersection

How can I use graphs to solve equations?

- One method to solve equations is to use graphs
- To solve $f(x) = a$
  - Plot the two graphs $y = f(x)$ and $y = a$ on your GDC
  - Find the points of intersections
  - The ***x*-coordinates** are the **solutions** of the equation
- To solve $f(x) = g(x)$
  - Plot the two graphs $y = f(x)$ and $y = g(x)$ on your GDC
  - Find the points of intersections
  - The ***x*-coordinates** are the **solutions** of the equation
- Using graphs makes it easier to see **how many solutions** an equation will have

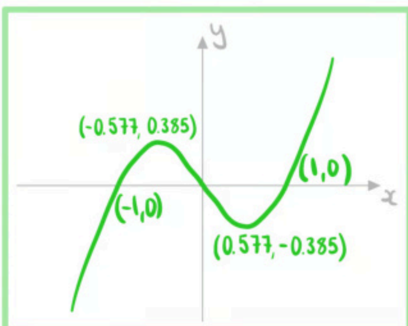
Worked example

Two functions are defined by

$$f(x) = x^3 - x \text{ and } g(x) = \frac{4}{x}.$$

a) Sketch the graph $y = f(x)$.

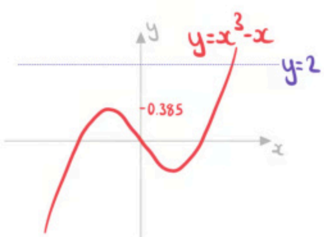
Use GDC to find max, min, intercepts



b) Write down the number of real solutions to the equation $x^3 - x = 2$.

Identify the number of intersections between

$y = x^3 - x$ and $y = 2$

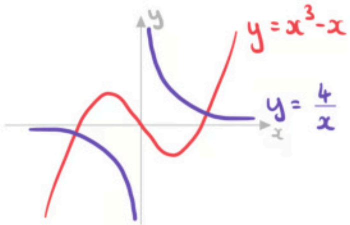


1 intersection

$\boxed{\text{1 solution}}$

c) Find the coordinates of the points where $y = f(x)$ and $y = g(x)$ intersect.

Use GDC to sketch both graphs



$\boxed{(-1.60, -2.50) \text{ and } (1.60, 2.50)}$

d) Write down the solutions to the equation $x^3 - x = \frac{4}{x}$.

Solutions to $x^3 - x = \frac{4}{x}$ are the x coordinates of the points of intersection.

$\boxed{x = -1.60 \text{ and } x = 1.60}$