DP IB Maths: AA SL

2.2 Quadratic Functions & Graphs

Contents

2.2.1 Quadratic Functions

Quadratic Functions & Graphs

What are the key features of quadratic graphs?

- A **quadratic** graph can be written in the form $y = ax^2 + bx + c$ where $a \neq 0$
- The value of a affects the shape of the curve
  - If a is **positive** the shape is **concave up** ∪
  - If a is **negative** the shape is **concave down** ∩
- The **y-intercept** is at the point $(0, c)$
- The **zeros or roots** are the solutions to $ax^2 + bx + c = 0$
  - These can be found by
    - Factorising
    - Quadratic formula
    - Using your GDC
  - These are also called the x-intercepts
  - There can be 0, 1 or 2 x-intercepts
    - This is determined by the value of the **discriminant**

- There is an **axis of symmetry** at $x = -\dfrac{b}{2a}$

  - This is given in your **formula booklet**
  - If there are two x-intercepts then the axis of symmetry goes through the midpoint of them
- The **vertex** lies on the axis of symmetry
  - It can be found by **completing the square**

  - The x-coordinate is $x = -\dfrac{b}{2a}$

  - The y-coordinate can be found using the GDC or by calculating y when $x = -\dfrac{b}{2a}$

  - If a is **positive** then the vertex is the **minimum point**
  - If a is **negative** then the vertex is the **maximum point**

What are the equations of a quadratic function?

- $f(x) = ax^2 + bx + c$
  - This is the **general form**
  - It clearly shows the y-intercept $(0, c)$
  - You can find the axis of symmetry by $x = -\dfrac{b}{2a}$
    - This is given in the formula booklet
- $f(x) = a(x - p)(x - q)$
  - This is the **factorised form**
  - It clearly shows the roots $(p, 0)$ & $(q, 0)$
  - You can find the axis of symmetry by $x = \dfrac{p + q}{2}$
- $f(x) = a(x - h)^2 + k$
  - This is the **vertex form**
  - It clearly shows the vertex (h, k)
  - The axis of symmetry is therefore $x = h$
  - It clearly shows how the function can be transformed from the graph $y = x^2$
    - Vertical stretch by scale factor a
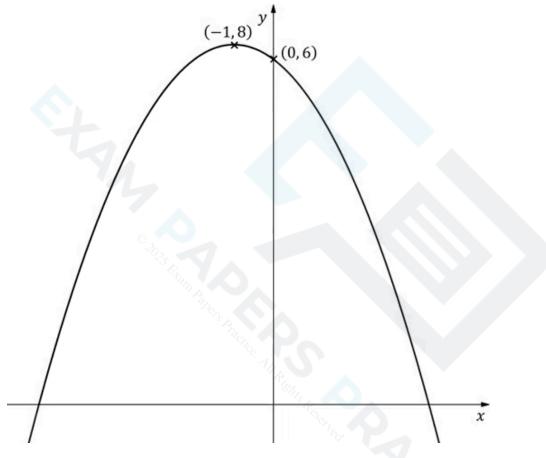   - Translation by vector $\begin{pmatrix} h \\ k \end{pmatrix}$

How do I find an equation of a quadratic?

- If you have the **roots** $x = p$ and $x = q$...
  - Write in **factorised form** $y = a(x - p)(x - q)$
  - You will need a third point to find the value of a
- If you have the **vertex** (h, k) then...
  - Write in **vertex form** $y = a(x - h)^2 + k$
  - You will need a second point to find the value of a
- If you have **three random points** (x_1, y_1), (x_2, y_2) & (x_3, y_3) then...
  - Write in the **general form** $y = ax^2 + bx + c$
  - Substitute the three points into the equation
  - Form and solve a system of three linear equations to find the values of a, b & c

✎ Worked example

The diagram below shows the graph of $y = f(x)$, where $f(x)$ is a quadratic function.

The intercept with the y-axis and the vertex have been labelled.

Write down an expression for $y = f(x)$.

We have the vertex so use $y = a(x-h)^2 + k$

Vertex $(-1, 8)$: $y = a(x - (-1))^2 + 8$

$$y = a(x + 1)^2 + 8$$

Substitute the second point

$x = 0, y = 6$: $6 = a(0 + 1)^2 + 8$

$$6 = a + 8$$

$$a = -2$$

$$y = -2(x + 1)^2 + 8$$

2.2.2 Factorising & Completing the Square

Factorising Quadratics

Why is factorising quadratics useful?

- Factorising gives **roots (zeroes or solutions)** of a quadratic
- It gives the **x-intercepts** when drawing the graph

How do I factorise a monic quadratic of the form $x^2 + bx + c$?

- A monic quadratic is a quadratic where the coefficient of the x^2 term is 1
- You might be able to spot the factors by **inspection**
    - Especially if c is a **prime number**
- Otherwise find two numbers m and n ..
    - A sum equal to b
        - $p + q = b$
    - A product equal to c
        - $pq = c$
- Rewrite bx as $mx + nx$
- Use this to factorise $x^2 + mx + nx + c$
- A shortcut is to write:
    - $(x + p)(x + q)$

How do I factorise a non-monic quadratic of the form $ax^2 + bx + c$?

- A non-monic quadratic is a quadratic where the coefficient of the x^2 term is not equal to 1
- If a, b & c have a common factor then first factorise that out to leave a quadratic with coefficients that have **no common factors**
- You might be able to spot the factors by **inspection**
    - Especially if a and/or c are **prime numbers**
- Otherwise find two numbers m and n ..
    - A sum equal to b
        - $m + n = b$
    - A product equal to ac
        - $mn = ac$
- Rewrite bx as $mx + nx$
- Use this to factorise $ax^2 + mx + nx + c$
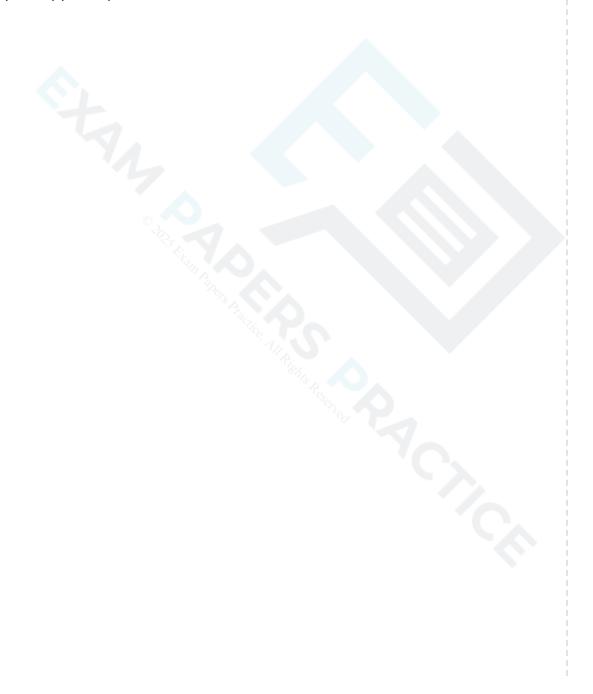- A shortcut is to write:
    - $$\frac{(ax + m)(ax + n)}{a}$$
    - Then factorise common factors from numerator to cancel with the a on the denominator

How do I use the difference of two squares to factorise a quadratic of the form $a^2x^2 - c^2$?

- The **difference of two squares** can be used when…
    - There is **no *x* term**
    - The **constant term is a negative**
- Square root the two terms a^2x^2 and c^2
- The two factors are the **sum of square roots** and the **difference of the square roots**
- A shortcut is to write:
    - $(ax + c)(ax - c)$

✏️ Worked example

Factorise fully:

a) $x^2 - 7x + 12$.

Find two numbers m and n such that

$m + n = b = -7$ $mn = c = 12$

$-4 + -3 = -7$ $-4 \times -3 = 12$

Split $-7x$ up and factorise   Shortcut

$x^2 - 4x - 3x + 12$ $(x+m)(x+n)$

$x(x-4) - 3(x-4)$ $\boxed{(x-3)(x-4)}$

$\boxed{(x-3)(x-4)}$

b) $4x^2 + 4x - 15$.

Find two numbers m and n such that

$m + n = b = 4$ $mn = ac = 4 \times -15 = -60$

$10 + -6 = 4$ $10 \times -6 = -60$

Split $4x$ up and factorise   Shortcut

$4x^2 + 10x - 6x - 15$ $\dfrac{(ax+m)(ax+n)}{a}$

$2x(2x+5) - 3(2x+5)$ $\dfrac{(4x+10)(4x-6)}{4}$

$\boxed{(2x-3)(2x+5)}$ $\dfrac{2(2x+5)\,2(2x-3)}{4}$

$\boxed{(2x-3)(2x+5)}$

c) $18 - 50x^2$.

Factorise the common factor

$2(9 - 25x^2)$

Use difference of two squares

$2(3 - 5x)(3 + 5x)$

Completing the Square

Why is completing the square for quadratics useful?

- Completing the square gives the **maximum/minimum** of a quadratic function
  - This can be used to define the **range** of the function
- It gives the **vertex** when drawing the graph
- It can be used to **solve quadratic equations**
- It can be used to derive the **quadratic formula**

How do I complete the square for a monic quadratic of the form $x^2 + bx + c$?

- **Half the value of b** and write $\left(x + \dfrac{b}{2}\right)^2$
  - This is because $\left(x + \dfrac{b}{2}\right)^2 = x^2 + bx + \dfrac{b^2}{4}$
- **Subtract the unwanted** $\dfrac{b^2}{4}$ term and **add on the constant c**
  - $\left(x + \dfrac{b}{2}\right)^2 - \dfrac{b^2}{4} + c$

How do I complete the square for a non-monic quadratic of the form $ax^2 + bx + c$?

- **Factorise out the a** from the **terms involving x**
  - $a\left(x^2 + \dfrac{b}{a}x\right) + x$
  - Leaving the c alone will **avoid working with lots of fractions**
- **Complete the square** on the quadratic term
  - **Half** $\dfrac{b}{a}$ and write $\left(x + \dfrac{b}{2a}\right)^2$
    - This is because $\left(x + \dfrac{b}{2a}\right)^2 = x^2 + \dfrac{b}{a}x + \dfrac{b^2}{4a^2}$
  - **Subtract the unwanted** $\dfrac{b^2}{4a^2}$ **term**
- **Multiply by a** and **add the constant c**
  - $a\left[\left(x + \dfrac{b}{2a}\right)^2 - \dfrac{b^2}{4a^2}\right] + c$
  - $a\left(x + \dfrac{b}{2a}\right)^2 - \dfrac{b^2}{4a} + c$

✏️ Worked example

Complete the square:

a) $x^2 - 8x + 3$.

Half b and subtract its square

$(x-4)^2 - 4^2 + 3$

$\boxed{(x-4)^2 - 13}$

b) $3x^2 + 12x - 5$.

Factorise the 3 from the x terms

$3(x^2 + 4x) - 5$

Complete the square on $x^2 + 4x$

$3\left((x+2)^2 - 2^2\right) - 5$

Simplify

$3\left((x+2)^2 - 4\right) - 5$

$3(x+2)^2 - 12 - 5$

$\boxed{3(x+2)^2 - 17}$

2.2.3 Solving Quadratics

Solving Quadratic Equations

How do I decide the best method to solve a quadratic equation?

- A quadratic equation is of the form $ax^2 + bx + c = 0$
- If it is a calculator paper then use your GDC to solve the quadratic
- If it is a non-calculator paper then...
    - you can always use the **quadratic formula**
    - you can **factorise** if it can be factorised with integers
    - you can always **complete the square**

How do I solve a quadratic equation by the quadratic formula?

- If necessary **rewrite** in the form $ax^2 + bx + c = 0$
- **Clearly identify** the values of a, b & c
- **Substitute** the values into the formula
    - $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
        - This is given in the **formula booklet**
- **Simplify** the solutions as much as possible

How do I solve a quadratic equation by factorising?

- **Factorise** to rewrite the quadratic equation in the form $a(x - p)(x - q) = 0$
- Set each factor to zero and **solve**
    - $x - p = 0 \Rightarrow x = p$
    - $x - q = 0 \Rightarrow x = q$

How do I solve a quadratic equation by completing the square?

- **Complete the square** to rewrite the quadratic equation in the form $a(x-h)^2 + k = 0$
- Get the squared term by itself

  - $(x-h)^2 = -\dfrac{k}{a}$

- If $\left(-\dfrac{k}{a}\right)$ is **negative** then there will be **no solutions**

- If $\left(-\dfrac{k}{a}\right)$ is **positive** then there will be **two values** for $x - h$

  - $x - h = \pm\sqrt{-\dfrac{k}{a}}$

- **Solve** for x

  - $x = h \pm \sqrt{-\dfrac{k}{a}}$

Worked example

Solve the equations:

a) $4x^2 + 4x - 15 = 0$.

This can be factorised

$(2x + 5)(2x - 3) = 0$

$2x + 5 = 0$ or $2x - 3 = 0$

$x = -\dfrac{5}{2}$ or $x = \dfrac{3}{2}$

b) $3x^2 + 12x - 5 = 0$.

This can not be factorised but $3x^2$ and $12x$ have a common
factor so complete the square

$3(x+2)^2 - 17 = 0$

$(x+2)^2 = \dfrac{17}{3}$ ← Rearrange

$x + 2 = \pm\sqrt{\dfrac{17}{3}}$ ← Remember $\pm$

$x = -2 \pm\sqrt{\dfrac{17}{3}}$

c) $7 - 3x - 5x^2 = 0$.

This can not be factorised so use formula

Formula booklet

Solutions of a quadratic equation	$ax^2 + bx + c = 0 \implies x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, $a \neq 0$

$a = -5 \quad b = -3 \quad c = 7$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(-5)(7)}}{2(-5)}$$

$$= \frac{3 \pm \sqrt{9 + 140}}{-10}$$

$$\boxed{x = -\frac{3 \pm \sqrt{149}}{10}}$$

2.2.4 Quadratic Inequalities

Quadratic Inequalities

What affects the inequality sign when rearranging a quadratic inequality?

- The inequality sign is **unchanged** by...
    - **Adding/subtracting** a term to both sides
    - **Multiplying/dividing** both sides by a **positive term**
- The inequality sign **flips** (< changes to >) when...
    - **Multiplying/dividing** both sides by a **negative term**

How do I solve a quadratic inequality?

- **STEP 1**: **Rearrange** the inequality into quadratic form with a **positive squared term**
    - $ax^2 + bx + c > 0$
    - $ax^2 + bx + c \geq 0$
    - $ax^2 + bx + c < 0$
    - $ax^2 + bx + c \leq 0$
- **STEP 2**: Find the **roots** of the quadratic equation
    - Solve $ax^2 + bx + c = 0$ to get x_1 and x_2 where $x_1 < x_2$
- **STEP 3**: **Sketch** a graph of the quadratic and label the roots
    - As the squared term is positive it will be **concave up** so "U" shaped
- **STEP 4**: **Identify** the **region** that satisfies the inequality
    - If you want the graph to be **above the x-axis** then choose the region to be the **two intervals outside** of the two roots
    - If you want the graph to be **below the x-axis** then choose the region to be the **interval between** the two roots
    - For $ax^2 + bx + c > 0$
        - The solution is $x < x_1$ **or** $x > x_2$
    - For $ax^2 + bx + c \geq 0$
        - The solution is $x \leq x_1$ **or** $x \geq x_2$
    - For $ax^2 + bx + c < 0$
        - The solution is $x_1 < x < x_2$
    - For $ax^2 + bx + c \leq 0$
        - The solution is $x_1 \leq x \leq x_2$

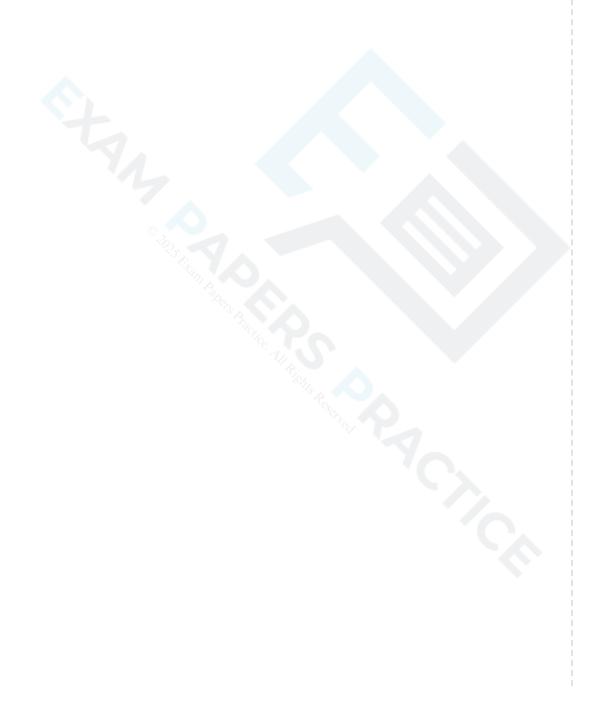### How do I solve a quadratic inequality of the form $(x - h)^2 < n$ or $(x - h)^2 > n$?

- The safest way is by following the steps above
    - Expand and rearrange

- A **common mistake** is writing $x - h < \pm\sqrt{n}$ or $x - h > \pm\sqrt{n}$
    - This is **NOT correct**!
- The correct solution to **$(x - h)^2 < n$** is

- $|x - h| < \sqrt{n}$ which can be written as $-\sqrt{n} < x - h < \sqrt{n}$
  - The **final solution** is $h - \sqrt{n} < x < h + \sqrt{n}$
- The correct solution to **(x – h)² > n** is
  - $|x - h| > \sqrt{n}$ which can be written as $x - h < -\sqrt{n}$ or $x - h > \sqrt{n}$
  - The **final solution** is $x < h - \sqrt{n}$ or $x > h + \sqrt{n}$

✏️ Worked example

Find the set of values which satisfy $3x^2 + 2x - 6 > x^2 + 4x - 2$.

STEP 1 : Rearrange

$(3x^2 + 2x - 6) - (x^2 + 4x - 2) > 0$ This way gives $a > 0$

$2x^2 - 2x - 4 > 0$

$x^2 - x - 2 > 0$ ← Divide by factor of 2

STEP 2 : Find the roots

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$
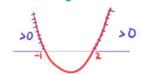
$x = 2$ or $x = -1$

STEP 3 : Sketch

STEP 4 : Identify region

>0 >0

$\boxed{x < -1 \quad \text{or} \quad x > 2}$

2.2.5 Discriminants

Discriminants

What is the discriminant of a quadratic function?

- The discriminant of a quadratic is denoted by the Greek letter Δ (upper case delta)
- For the quadratic function the discriminant is given by
  - $\Delta = b^2 - 4ac$
    - This is given in the **formula booklet**
- The discriminant is the expression that is square rooted in the **quadratic formula**

How does the discriminant of a quadratic function affect its graph and roots?

- If Δ > 0 then $\sqrt{b^2 - 4ac}$ and $-\sqrt{b^2 - 4ac}$ are **two distinct values**
  - The equation $ax^2 + bx + c = 0$ has **two distinct real solutions**
  - The graph of $y = ax^2 + bx + c$ has **two distinct real roots**
    - This means the graph **crosses** the x-axis **twice**
- If Δ = 0 then $\sqrt{b^2 - 4ac}$ and $-\sqrt{b^2 - 4ac}$ are **both zero**
  - The equation $ax^2 + bx + c = 0$ has **one repeated real solution**
  - The graph of $y = ax^2 + bx + c$ has **one repeated real root**
    - This means the graph **touches** the x-axis at **exactly one point**
    - This means that the **x-axis** is a **tangent** to the graph
- If Δ < 0 then $\sqrt{b^2 - 4ac}$ and $-\sqrt{b^2 - 4ac}$ are **both undefined**
  - The equation $ax^2 + bx + c = 0$ has **no real solutions**
  - The graph of $y = ax^2 + bx + c$ has **no real roots**
    - This means the graph **never touches** the x-axis
    - This means that graph is **wholly above** (or **below**) the x-axis

Forming equations and inequalities using the discriminant

- Often at least one of the coefficients of a quadratic is **unknown**
    - Questions usually use the letter *k* for the unknown constant
- You will be given a fact about the quadratic such as:
    - The **number of solutions** of the equation
    - The **number of roots** of the graph

- To find the **value or range of values** of *k*
    - Find an **expression for the discriminant**
        - Use $\Delta = b^2 - 4ac$
    - Decide whether $\Delta > 0$, $\Delta = 0$ or $\Delta < 0$
        - If the question says there are **real roots** but does not specify how many then use $\Delta \geq 0$
    - **Solve** the resulting equation or inequality

✎ Worked example

A function is given by $f(x) = 2kx^2 + kx - k + 2$, where k is a constant. The graph of $y = f(x)$ has two distinct real roots.

a) Show that $9k^2 - 16k > 0$.

Two distinct real roots $\Rightarrow$ $\Delta > 0$

Formula booklet

Discriminant	$\Delta = b^2 - 4ac$

$a = 2k \quad b = k \quad c = (-k + 2)$

$\Delta = k^2 - 4(2k)(-k+2)$

$\quad = k^2 + 8k^2 - 16k$

$\quad = 9k^2 - 16k$

$\Delta > 0 \quad \Rightarrow \quad 9k^2 - 16k > 0$

b) Hence find the set of possible values of k.

Solve the inequality

$9k^2 - 16k = 0$

$k(9k - 16) = 0$

$k = 0 \quad or \quad k = \dfrac{16}{9}$

$k < 0 \quad or \quad k > \dfrac{16}{9}$