# DP IB Maths: AI SL

## 2.2 Further Functions & Graphs

### Contents

## 2.2.1 Functions

# Language of Functions

### What is a mapping?

- A **mapping transforms** one set of values (**inputs**) into another set of values (**outputs**)
- Mappings can be:
  - **One-to-one**
    - Each input gets mapped to **exactly one unique** output
    - No two inputs are mapped to the same output
    - For example: A mapping that cubes the input
  - **Many-to-one**
    - Each input gets mapped to **exactly one** output
    - Multiple inputs can be mapped to the same output
    - For example: A mapping that squares the input
  - **One-to-many**
    - An input can be mapped to **more than one** output
    - No two inputs are mapped to the same output
    - For example: A mapping that gives the numbers which when squared equal the input
  - **Many-to-many**
    - An input can be mapped to **more than one** output
    - Multiple inputs can be mapped to the same output
    - For example: A mapping that gives the factors of the input

### What is a function?

- A **function** is a mapping between two sets of numbers where **each input** gets mapped to **exactly one output**
  - The output does not need to be unique

- **One-to-one** and **many-to-one** mappings are functions
- A mapping is a function if its graph passes the **vertical line test**
  - Any **vertical line** will intersect with the graph **at most once**

### What notation is used for functions?

- Functions are denoted using letters (such as $f$, $v$, $g$, etc)
  - A function is followed by a variable in a bracket
  - This shows the input for the function
  - The letter $f$ is used most commonly for functions and will be used for the remainder of this revision note
- $f(x)$ represents an expression for the value of the function $f$ when evaluated for the variable $x$
- Function notation gets rid of the need for words which makes it **universal**
  - $f = 5$ when $x = 2$ can simply be written as $f(2) = 5$

### What are the domain and range of a function?

- The **domain** of a function is the set of values that are used as **inputs**
- A domain should be stated with a function
  - If a domain is not stated then it is assumed the domain is all the real values which would work as inputs for the function
  - Domains are expressed in terms of the input
    - $x \leq 2$
- The **range** of a function is the set of values that are given as **outputs**
  - The range depends on the domain
  - Ranges are expressed in terms of the output
    - $f(x) \geq 0$
- To graph a function we use the **inputs as the *x*-coordinates** and the **outputs as the *y*-coordinates**
  - $f(2) = 5$ corresponds to the coordinates (2, 5)
- Graphing the function can help you visualise the range
- Common sets of numbers have special symbols:
  - $\mathbb{R}$ represents all the real numbers that can be placed on a number line
    - $x \in \mathbb{R}$ means $x$ is a real number
  - $\mathbb{Q}$ represents all the rational numbers $\frac{a}{b}$ where *a* and *b* are integers and $b \neq 0$
  - $\mathbb{Z}$ represents all the integers (positive, negative and zero)
    - $\mathbb{Z}^+$ represents positive integers
  - $\mathbb{N}$ represents the natural numbers (0,1,2,3...)

## Worked example

For the function $f(x) = x^3 + 1,\ 2 \le x \le 10$:

a) write down the value of $f(7)$.

Substitute $x = 7$

$f(7) = 7^3 + 1$

$\boxed{f(7) = 344}$

b) find the range of $f(x)$.

Find the values of $x^3+1$ when $2 \le x \le 10$

$2 \le x \le 10$

$8 \le x^3 \le 1000$

$9 \le x^3+1 \le 1001$

$\boxed{9 \le f(x) \le 1001}$

# Inverse Functions

## What is an inverse function?

- **Only one-to-one** functions have inverses
- A function has an inverse if its graph passes the **horizontal line test**
  - Any **horizontal line** will intersect with the graph **at most once**
- Given a function $f(x)$ we denote the inverse function as $f^{-1}(x)$
- An inverse function **reverses the effect** of a function
  - $f(2) = 5$ means $f^{-1}(5) = 2$
- Inverse functions are used to solve equations
  - The solution of $f(x) = 5$ is $x = f^{-1}(5)$

## What are the connections between a function and its inverse function?

- The **domain of a function** becomes the **range of its inverse**
- The **range of a function** becomes the **domain of its inverse**
- The graph of $y = f^{-1}(x)$ is a **reflection** of the graph $y = f(x)$ in the line $y = x$
  - Therefore solutions to $f(x) = x$ or $f^{-1}(x) = x$ will also be solutions to $f(x) = f^{-1}(x)$
    - There could be other solutions to $f(x) = f^{-1}(x)$ that don't lie on the line $y = x$

## Worked example

For the function $f(x) = x^3 + 1,\ \ 2 \le x \le 10$:

a) write down the range of the inverse function, $f^{-1}(x)$.

The range of $f^{-1}(x)$ is the domain of $f(x)$

$\boxed{2 \le f^{-1}(x) \le 10}$

b) find the value of $f^{-1}(217)$.

If $x = f^{-1}(a)$ then $f(x) = a$

$x = f^{-1}(217)$

$f(x) = 217$

$x^3 + 1 = 217$ Use definition of function

$x^3 = 216$ Subtract 1

$x = 6$ Cube root

$\boxed{f^{-1}(217) = 6}$

## Piecewise Functions

### What are piecewise functions?

- **Piecewise functions** are defined by different functions depending on which interval the input is in
  - E.g. $f(x) = \begin{cases} x+1 & x \le 5 \\ 2x-4 & 5 < x < 10 \end{cases}$
- The intervals for the individual functions **cannot overlap**
- To evaluate a piecewise function for a particular value $x = k$
  - Find which interval includes $k$
  - Substitute $x = k$ into the corresponding function

### Worked example

For the piecewise function

$$f(x) = \begin{cases} 2x-5 & -10 \le x \le 10 \\ 3x+1 & x > 10 \end{cases},$$

a) find the values of $f(0),\ f(10),\ f(20)$.

Identify the correct function to use

$x = 0$ is in $-10 \le x \le 10 \Rightarrow f(0) = 2(0) - 5 = -5$

$x = 10$ is in $-10 \le x \le 10 \Rightarrow f(10) = 2(10) - 5 = 15$

$x = 20$ is in $x > 10 \Rightarrow f(20) = 3(20) + 1 = 61$

$\boxed{f(0) = -5 \quad f(10) = 15 \quad f(20) = 61}$

b) state the domain.

Domain is the set of inputs

$-10 \le x \le 10$ and $x > 10$

$\boxed{x \ge -10}$

## 2.2.2 Graphing Functions

## Graphing Functions

### How do I graph the function *y* = *f*(*x*)?

- A point $(a, b)$ lies on the graph $y = f(x)$ if $f(a) = b$
- The **horizontal axis** is used for the **domain**
- The **vertical axis** is used for the **range**
- You will be able to graph some functions by hand
- For some functions you will need to use your GDC
- You might be asked to graph the **sum** or **difference** of two functions
  - Use your GDC to graph $y = f(x) + g(x)$ or $y = f(x) - g(x)$
  - Just type the functions into the graphing mode

### What is the difference between "draw" and "sketch"?

- If asked to sketch you should:
  - Show the general shape
  - Label any key points such as the intersections with the axes
  - Label the axes
- If asked to draw you should:
  - Use a pencil and ruler
  - Draw to scale
  - Plot any points **accurately**
  - Join points with a straight line or smooth curve
  - Label any key points such as the intersections with the axes
  - Label the axes

### How can my GDC help me sketch/draw a graph?

- You use your GDC to plot the graph
  - Check the scales on the graph to make sure you see the full shape
- Use your GDC to find any key points
- Use your GDC to check specific points to help you plot the graph

## Key Features of Graphs

### What are the key features of graphs?

- You should be familiar with the following key features and know how to use your GDC to find them
- Local minimums/maximums
  - These are points where the graph has a minimum/maximum for a small region
  - They are also called **turning points**
    - This is where the graph changes its direction between upwards and downwards directions
  - A graph can have multiple local minimums/maximums
  - A local minimum/maximum is not necessarily the minimum/maximum of the whole graph
    - This would be called the **global** minimum/maximum
  - For quadratic graphs the minimum/maximum is called the **vertex**
- Intercepts
  - $y$ – intercepts are where the graph crosses the $y$-axis
    - At these points $x = 0$
  - $x$ – intercepts are where the graph crosses the $x$-axis
    - At these points $y = 0$
    - These points are also called the **zeros of the function** or **roots of the equation**
- Symmetry
  - Some graphs have lines of symmetry
    - A quadratic will have a vertical line of symmetry
- Asymptotes
  - These are lines which the graph will get closer to but not cross
  - These can be horizontal or vertical
    - Exponential graphs have horizontal asymptotes
    - Graphs of variables which vary inversely can have vertical and horizontal asymptotes

## Worked example

Two functions are defined by

$$f(x) = x^2 - 4x - 5 \text{ and } g(x) = 2 + \frac{1}{x+1}.$$

a) Draw the graph $y = f(x)$.

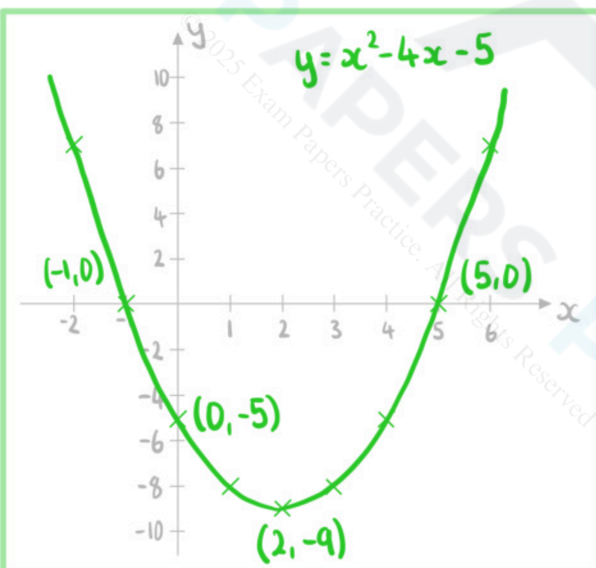
Draw means accurately

Use GDC to find vertex, roots and y-intercepts

Vertex = (2, -9)

Roots = (-1, 0) and (5, 0)

y-intercept = (0, -5)




b) Sketch the graph $y = g(x)$.

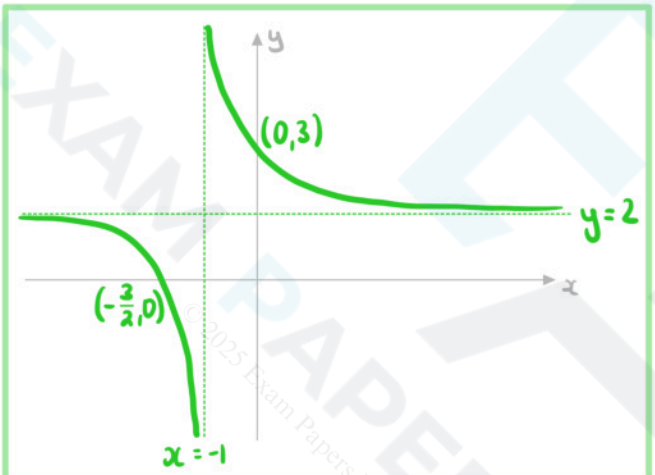
Sketch means rough but showing key points

Use GDC to find $x$ and $y$-intercepts and asymptotes

$x$-intercept $= \left(-\frac{3}{2}, 0\right)$

$y$-intercept $= (0, 3)$

Asymptotes : $x = -1$ and $y = 2$



# Intersecting Graphs

## How do I find where two graphs intersect?

- Plot both graphs on your GDC
- Use the intersect function to find the intersections
- Check if there is more than one point of intersection

## How can I use graphs to solve equations?

- One method to solve equations is to use graphs
- To solve $f(x) = a$
  - Plot the two graphs $y = f(x)$ and $y = a$ on your GDC
  - Find the points of intersections
  - The ***x*-coordinates** are the **solutions** of the equation
- To solve $f(x) = g(x)$
  - Plot the two graphs $y = f(x)$ and $y = g(x)$ on your GDC
  - Find the points of intersections
  - The ***x*-coordinates** are the **solutions** of the equation
- Using graphs makes it easier to see **how many solutions** an equation will have

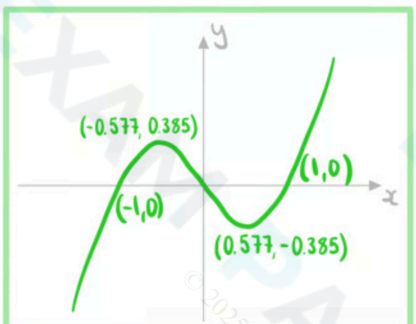
## Worked example

Two functions are defined by

$$f(x) = x^3 - x \text{ and } g(x) = \frac{4}{x}.$$

a) Sketch the graph $y = f(x)$.

Use GDC to find max, min, intercepts




b) Write down the number of real solutions to the equation $x^3 - x = 2$.

Identify the number of intersections between

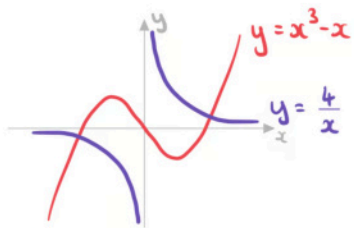
$y = x^3 - x$ and $y = 2$


1 intersection

$\boxed{1 \text{ solution}}$

c) Find the coordinates of the points where $y = f(x)$ and $y = g(x)$ intersect.

Use GDC to sketch both graphs




$\boxed{(-1.60, -2.50) \text{ and } (1.60, 2.50)}$

d)

Write down the solutions to the equation $x^3 - x = \frac{4}{x}$.

Solutions to $x^3 - x = \frac{4}{x}$ are the $x$ coordinates of the points of intersection.

$\boxed{x = -1.60 \text{ and } x = 1.60}$

## 2.2.3 Properties of Graphs

### Quadratic Functions & Graphs

**What are the key features of quadratic graphs?**

- A **quadratic** graph is of the form $y = ax^2 + bx + c$ where $a \neq 0$.
- The value of *a* affects the shape of the curve
  - If *a* is positive the shape is $\cup$
  - If *a* is negative the shape is $\cap$
- The ***y*-intercept** is at the point (0, c)
- The **zeros or roots** are the solutions to $ax^2 + bx + c = 0$
  - These can be found using your GDC or the quadratic formula
  - These are also called the *x*-intercepts
  - There can be 0, 1 or 2 *x*-intercepts
- There is an **axis of symmetry** at $x = -\dfrac{b}{2a}$
  - This is given in your **formula booklet**
  - If there are two *x*-intercepts then the axis of symmetry goes through the midpoint of them
- The **vertex** lies on the axis of symmetry
  - The *x*-coordinate is $-\dfrac{b}{2a}$
  - The *y*-coordinate can be found using the GDC or by calculating *y* when $x = -\dfrac{b}{2a}$
  - If ***a* is positive** then the vertex is the **minimum** point
  - If ***a* is negative** then the vertex is the **maximum** point

## Worked example

a) Write down the equation of the axis of symmetry for the graph $y = 4x^2 - 4x - 3$.

Formula booklet

| | |
|---|---|
| Axis of symmetry of the graph of a quadratic function | $f(x) = ax^2 + bx + c \Rightarrow$ axis of symmetry is $x = -\frac{b}{2a}$ |

$a = 4 \quad b = -4 \quad c = -3$

$x = -\frac{-4}{2(4)}$

$\boxed{x = \frac{1}{2}}$

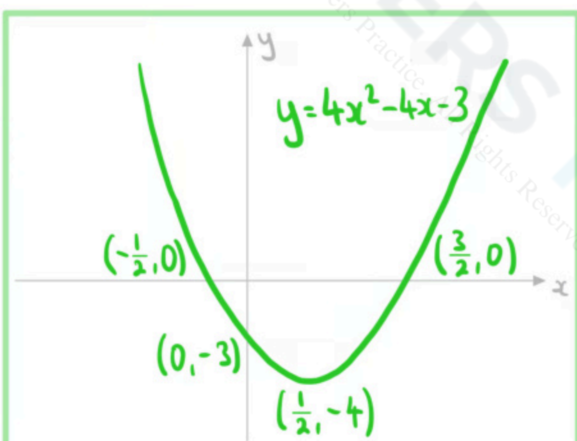
b) Sketch the graph $y = 4x^2 - 4x - 3$.

Use GDC to find vertex, roots and y-intercepts

Vertex : $(\frac{1}{2}, -4)$

Roots : $(-\frac{1}{2}, 0)$ and $(\frac{3}{2}, 0)$

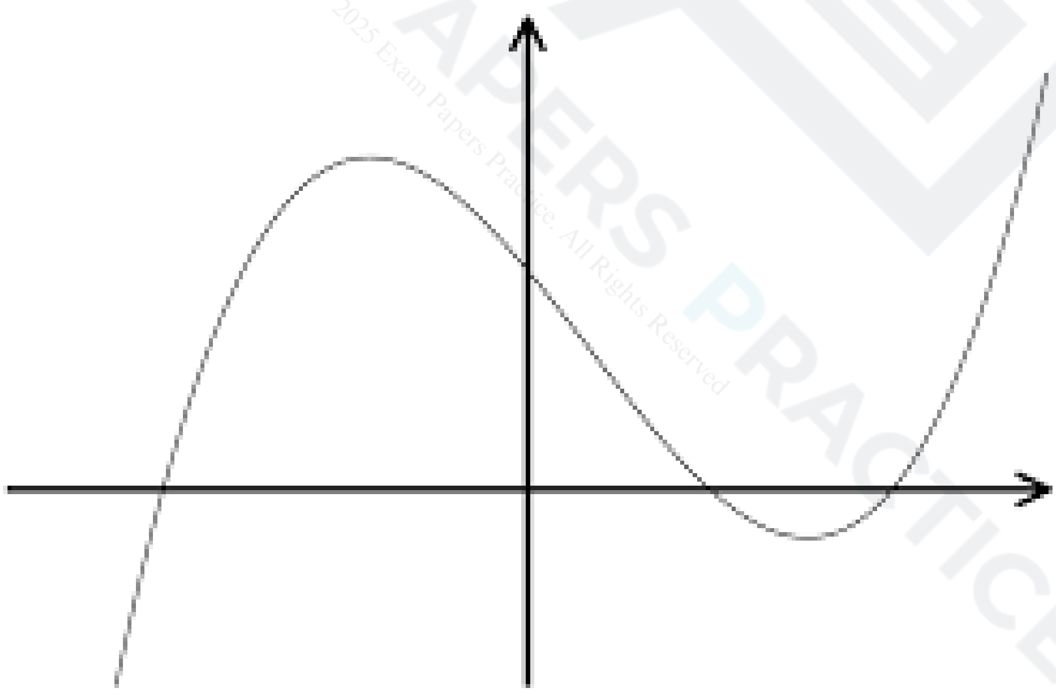
y-intercept : $(0, -3)$



## Cubic Functions & Graphs

### What are the key features of cubic graphs?

- A **cubic** graph is of the form $y = ax^3 + bx^2 + cx + d$ where $a \neq 0$.
- The value of *a* affects the shape of the curve
  - If ***a* is positive** the graph goes from **bottom left to top right**
  - If ***a* is negative** the graph goes from **top left to bottom right**
- The ***y*-intercept** is at the point (0, *d*)
- The **zeros or roots** are the solutions to $ax^3 + bx^2 + cx + d = 0$
  - These can be found using your GDC
  - These are also called the *x*-intercepts
  - There can be 1, 2 or 3 *x*-intercepts
    - There is always at least 1
- There are either **0 or 2 local minimums/maximums**
  - If there are 0 then the curve is **monotonic** (always increasing or always decreasing)
  - If there are 2 then one is a local minimum and one is a local maximum



### Worked example

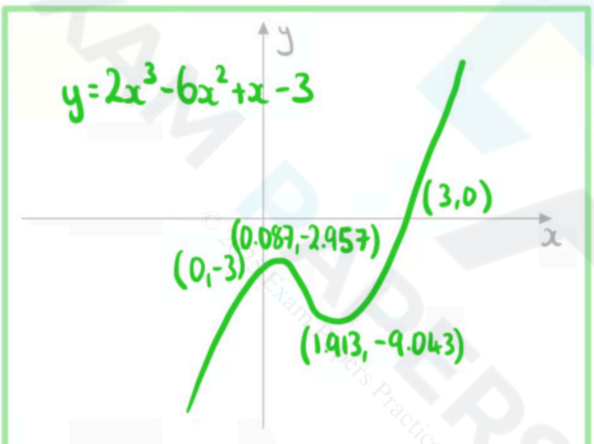
Sketch the graph $y = 2x^3 - 6x^2 + x - 3$.

Use GDC to find min, max, roots and y-intercept

Min = (1.913, -9.043)    Max = (0.087, -2.957)

Root = (3,0)

y-intercept = (0, -3)



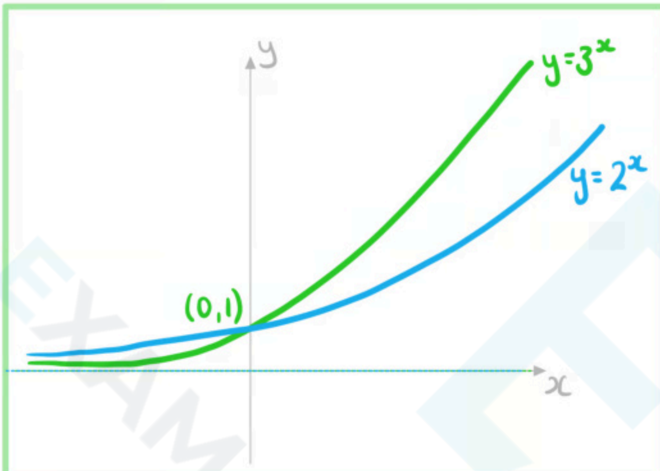
## Exponential Functions & Graphs

### What are the key features of exponential graphs?

- An **exponential** graph is of the form
  - $y = ka^x + c$ or $y = ka^{-x} + c$ where $a > 0$
  - $y = ke^{rx} + c$
    - Where e is the mathematical constant 2.718…
- The ***y*-intercept** is at the point (0, $k + c$)
- There is a **horizontal asymptote** at $y = c$
- The value of $k$ determines whether the graph is **above or below the asymptote**
  - If ***k* is positive** the graph is **above the asymptote**
    - So the range is $y > c$
  - If ***k* is negative** the graph is **below the asymptote**
    - So the range is $y < c$
- The coefficient of $x$ and the constant $k$ determine whether the graph is **increasing or decreasing**
  - If the coefficient of $x$ and $k$ have the **same sign** then **graph is increasing**
  - If the coefficient of $x$ and $k$ have **different signs** then the **graph is decreasing**
- There is at **most 1 root**
  - It can be found using your GDC

## Worked example

a) On the same set of axes sketch the graphs $y = 2^x$ and $y = 3^x$. Clearly label each graph.




b) Sketch the graph $y = 2\mathrm{e}^{-3x} + 1$.

Use GDC to find intercept and asymptote

y-intercept : (0, 3)

Asymptote : y = 1

## Sinusoidal Functions & Graphs

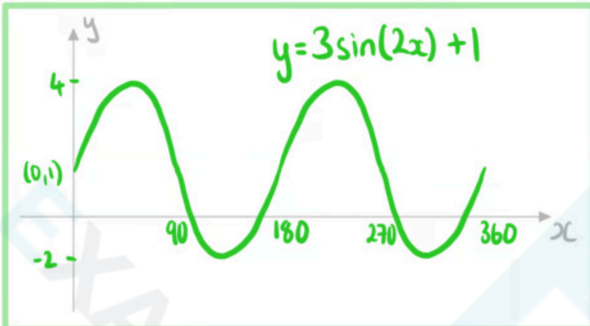
### What are the key features of sinusoidal graphs?

- A **sinusoidal** graph is of the form
  - $y = a\sin(bx) + d$
  - $y = a\cos(bx) + d$
- The ***y*-intercept** is at the point
  - $(0, d)$ for $y = a\sin(bx) + d$
  - $(0, a + d)$ for $y = a\cos(bx) + d$
- The **period** of the graph is the length of the interval of a full cycle
  - This is $\frac{360°}{b}$
- The **maximum value** is $y = a + d$
- The **minimum value** is $y = -a + d$
- The **principal axis** is the horizontal line halfway between the maximum and minimum values
  - This is $y = d$
- The **amplitude** is the vertical distance from the principal axis to the maximum value
  - This is $a$

## Worked example

a) Sketch the graph $y = 3\sin(2x) + 1$ for the values $0 \le x \le 360$.

Use GDC to find max and min




b) State the equation of the principal axis of the curve.

Principal axis is in middle of maximum and minimum points

$$\frac{4 + {}^{-}2}{2} = 1$$

$\boxed{y=1}$

c) State the period and amplitude.

Period is how often it repeats

$$\frac{360}{2} = 180$$

$\boxed{\text{Period} = 180°}$

Amplitude is distance from principal axis to maximum or minimum

$4 - 1 = 1 - {}^{-}2 = 3$

$\boxed{\text{Amplitude} = 3}$