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10.1 Rotational Dynamics

Engineering Physics

AQA A Level Revision Notes

A Level Physics AQA

11.1 Rotational Dynamics

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11.1.1 Rotational Motion

Angular Displacement, Velocity & Acceleration

- A rigid rotating body can be described using the following properties:
  - Angular displacement
  - Angular velocity
  - Angular acceleration
- These properties can be inferred from the properties of objects moving in a straight line combined with the geometry of circles and arcs

Angular Displacement

- Angular displacement is defined as:

The change in angle through which a rigid body has rotated relative to a fixed point

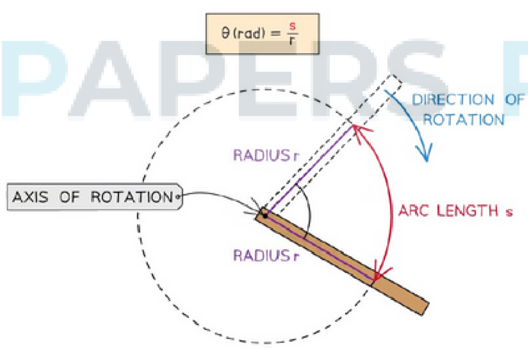
- Angular displacement is measured in radians

Angular displacement to linear displacement

- The linear displacement s at any point along a segment that is in rotation can be calculated using:

$$s = r\theta$$

- Where:
  - θ = angular displacement, or change in angle (radians)
  - s = length of the arc, or the linear distance travelled along a circular path (m)
  - r = radius of a circular path, or distance from the axis of rotation (m)



An angle in radians, subtended at the centre of a circle, is the arc length divided by the radius of the circle

Angular Velocity

- The angular velocity ω of a rigid rotating body is defined as:

The rate of change in angular displacement with respect to time

- Angular velocity is measured in rad s^{-1}

- This can be expressed as an equation:

$$\omega = \frac{\Delta\theta}{\Delta t}$$

- Where:
  - ω = angular velocity (rad s^{-1})
  - $\Delta\theta$ = angular displacement (rad)
  - Δt = change in time (s)

Angular velocity to linear velocity

- The linear speed v is related to the angular speed ω by the equation:

$$v = r\omega$$

- Where:
  - v = linear speed (m s^{-1})
  - r = distance from the axis of rotation (m)

- Taking the angular displacement of a complete cycle as 2π, angular velocity ω can also be expressed as:

$$\omega = \frac{v}{r} = 2\pi f = \frac{2\pi}{T}$$

- Rearranging gives the expression for linear speed:

$$v = 2\pi f r = \frac{2\pi r}{T}$$

- Where:
  - f = frequency of the rotation (Hz)
  - T = time period of the rotation (s)

Angular Acceleration

- Angular acceleration α is defined as

The rate of change of angular velocity with time

- Angular acceleration is measured in rad s^{-2}
- This can be expressed as an equation:

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

- Where:
  - α = angular acceleration (rad s^{-2})
  - $\Delta\omega$ = change in angular velocity, or $\Delta\omega = \omega_f - \omega_i$ (rad s^{-1})
  - Δt = change in time (s)

Angular acceleration to linear acceleration

- Using the definition of angular velocity ω with the equation for angular acceleration α gives:

$$\Delta\omega = \frac{\Delta v}{r}$$

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{\Delta v}{r\Delta t} = \frac{a}{r}$$

- Rearranging gives the expression for linear acceleration:

$$a = r\alpha$$

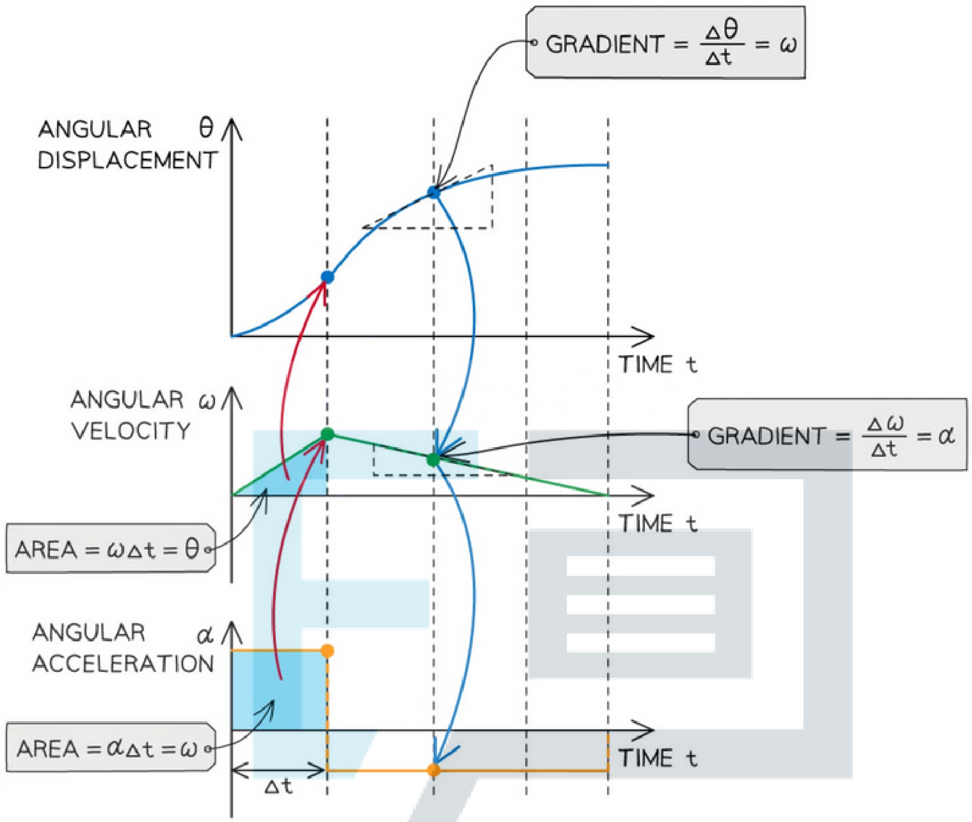
- Where:
  - a = linear acceleration (m s^{-2})
  - r = distance from the axis of rotation (m)
  - Δv = change in linear velocity, or $\Delta v = v - u$ (m s^{-1})

Exam Tip

While there are many similarities between the angular quantities used in this topic and the angular quantities used in the circular motion topic, make sure you are clear on the distinctions between the two, for example, **angular** acceleration and **centripetal** acceleration are **not** the same thing!

Graphs of Rotational Motion

Graphs of rotational motion can be interpreted in the same way as linear motion graphs



Graphs of angular displacement, angular velocity and angular acceleration

- Angular displacement, θ is equal to...
  - The **area under** the angular velocity-time graph
- Angular velocity, ω is equal to...
  - The **gradient** of the angular displacement-time graph
  - The **area under** the angular acceleration-time graph
- Angular acceleration, α is equal to...
  - The **gradient** of the angular velocity-time graph

Summary of linear and angular variables

Variable	Linear	Angular
displacement	$s = r\theta$	$\theta = \frac{s}{r}$
velocity	$v = r\omega$	$\omega = \frac{v}{r}$

acceleration	$a = r\alpha$	$\alpha = \frac{a}{r}$

Equations for Uniform Angular Acceleration

- The kinematic equations of motion for uniform linear acceleration can also be re-written for rotational motion
- The four kinematic equations for uniform **linear** acceleration are

$$v = u + at$$

$$s = ut + \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as$$

$$s = \frac{(u + v)t}{2}$$

- This leads to the four kinematic equations for uniform **rotational** acceleration

$$\omega_2 = \omega_1 + \alpha t$$

$$\Delta\theta = \omega_1 t + \frac{1}{2}\alpha t^2$$

$$\omega_2^2 = \omega_1^2 + 2\alpha\Delta\theta$$

$$\Delta\theta = \frac{(\omega_1 + \omega_2)t}{2}$$

- The five linear variables have been swapped for the rotational equivalents, as shown in the table below

Variable	Linear	Rotational
displacement	s	θ
initial velocity	u	ω_1
final velocity	v	ω_2
acceleration	a	α
time	t	t

Worked Example

The turntable of a record player is spinning at an angular velocity of 45 RPM just before it is turned off. It then decelerates at a constant rate of 0.8 rad s^{-2}.

Determine the number of rotations the turntable completes before coming to a stop.

Answer:

Step 1: List the known quantities

- Initial angular velocity, ω_1 = 45 RPM
- Final angular velocity, ω_2 = 0
- Angular acceleration, α = –0.8 rad s^{-2}
- Angular displacement, $\Delta\theta$ = ?

Step 2: Convert the angular velocity from RPM to rad s^{-1}

- One revolution corresponds to 2π radians, and RPM = revolutions per minute, so

$$\omega = 2\pi f \text{ and } f = \frac{RPM}{60} \text{ (to convert to seconds)}$$

$$\omega_1 = \frac{2\pi \times \text{RPM}}{60} = \frac{2\pi \times 45}{60} = \frac{3\pi}{2} \text{ rad s}^{-1}$$

Step 3: Select the most appropriate kinematic equation

- We know the values of ω_1, ω_2 and α, and we are looking for angular displacement θ, so the best equation to use would be

$$\omega_2^2 = \omega_1^2 + 2\alpha\Delta\theta$$

Step 4: Rearrange and calculate the angular displacement $\Delta\theta$

$$0 = \omega_1^2 + 2\alpha\Delta\theta$$

$$-2\alpha\Delta\theta = \omega_1^2$$

$$\Delta\theta = \frac{\omega_1^2}{-2\alpha} = \frac{\left(\frac{3\pi}{2}\right)^2}{-2 \times -0.8}$$

Angular displacement, $\Delta\theta$ = 13.88 rad

Step 5: Determine the number of rotations in $\Delta\theta$

- There are 2π radians in 1 rotation

- Therefore, the number of rotations = $\frac{13.88}{2\pi}$ = 2.2
- This means the turntable spins 2.2 times before coming to a stop

11.1.3 Torque

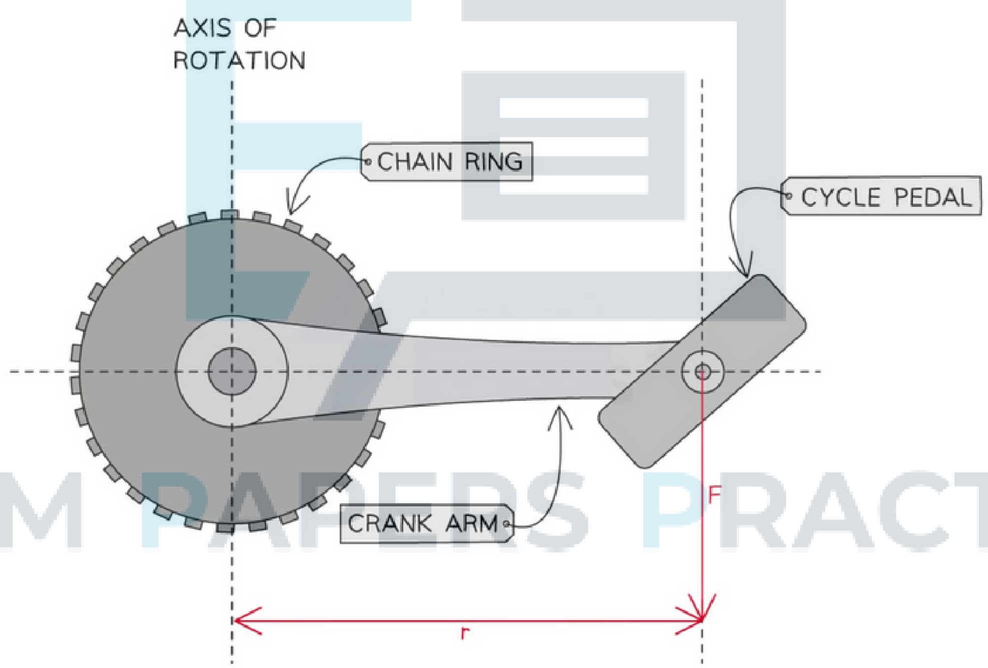
Torque

- The change in rotational motion due to a turning force is called **torque**
- The torque of a force *F* about an axis is given by

$$\tau = Fr$$

- Where:
  - τ = torque (N m)
  - F = applied force (N)
  - r = perpendicular distance between the axis of rotation and the line of action of the force (m)

Torque of a perpendicular force



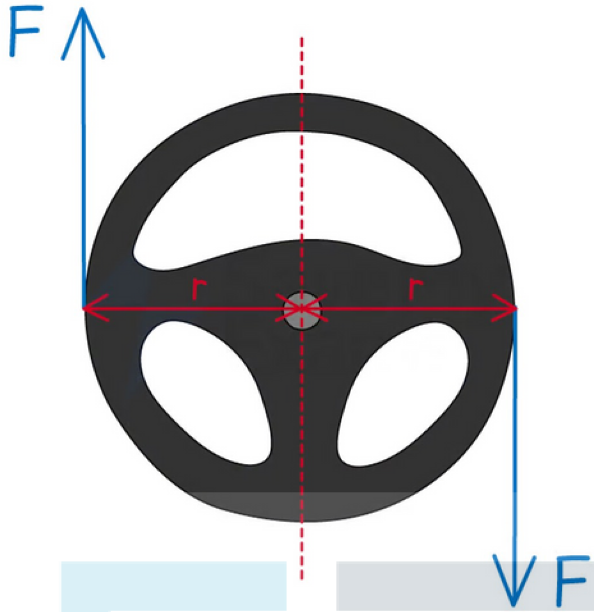
The torque applied by a cyclist on a bicycle pedal can be determined using the magnitude of the applied force, from the cyclist, and the distance between the line of action of the force and the axis of rotation, the length of the crank arm r

Torque of a Couple

- When applied to a couple (2 forces), torque can be described as

The sum of the moments produced by each of the forces in the couple

Torque of a couple on a steering wheel



The steering wheel is in rotational equilibrium since the resultant force and resultant torque are zero. This means it does not have linear or angular acceleration.

- For example, the torque provided by a couple on a steering wheel of radius *r* is

$$\tau = (F \times r) + (F \times r) = 2Fr$$

- Therefore, the torque of a couple is equal to **double** the magnitude of the torque of the individual forces
- The forces are equal and act in opposite directions
  - Therefore, couples produce a resultant force of **zero**
- Due to Newton's Second law (***F*** = ***ma***), the steering wheel does **not** accelerate
  - In other words, when the force is applied, the steering wheel rotates with a **constant** angular speed but remains in the same location

Worked Example

A grinding wheel is used to sharpen the edges of pieces of wood in a school workshop. A piece of wood is forced against the edge of the wheel so that the tangential force on the wheel is a steady 12.0 N as the wheel rotates. The diameter of the wheel is 0.13 m.

Calculate the torque on the grinding wheel, giving an appropriate unit

Answer:

Step 1: State the equation for torque

$$\tau = Fr$$

Step 2: Calculate the perpendicular distance from the axis of rotation

- This is the radius of the wheel since the rotation is from the centre of the wheel

$$r = \frac{0.13}{2} = 0.065 \text{ m}$$

Step 3: Substitute the values

$$\tau = 12.0 \times 0.065 = 0.78$$

Step 4: State the appropriate unit

$$\tau = 0.78 \text{ N m}$$

Exam Tip

The terminology in this section can get confusing. For example, a moment is not a 'turning force' - the turning force is only part of the moment, the moment is the **effect** that the turning force has on the system when applied at a **distance** from a turning point, or pivot.

This is often linked with content from the moments section of the course.

Ultimately, when you carry out calculations, make sure you can identify

- The magnitude of the **applied force**
- The **perpendicular distance** between the force and the turning point (along the line of action)

When considering an object in rotational equilibrium, choosing certain points can simplify calculations of resultant torque. Remember you can choose **any** point, not just the axis of rotation.

To simplify your calculation, choose a point where the torque of (most of) the forces are unknown, or when you need to determine where the resultant torque is **zero**. To do this, choose a point through which the lines of action of the forces pass.

11.1.4 Moment of Inertia

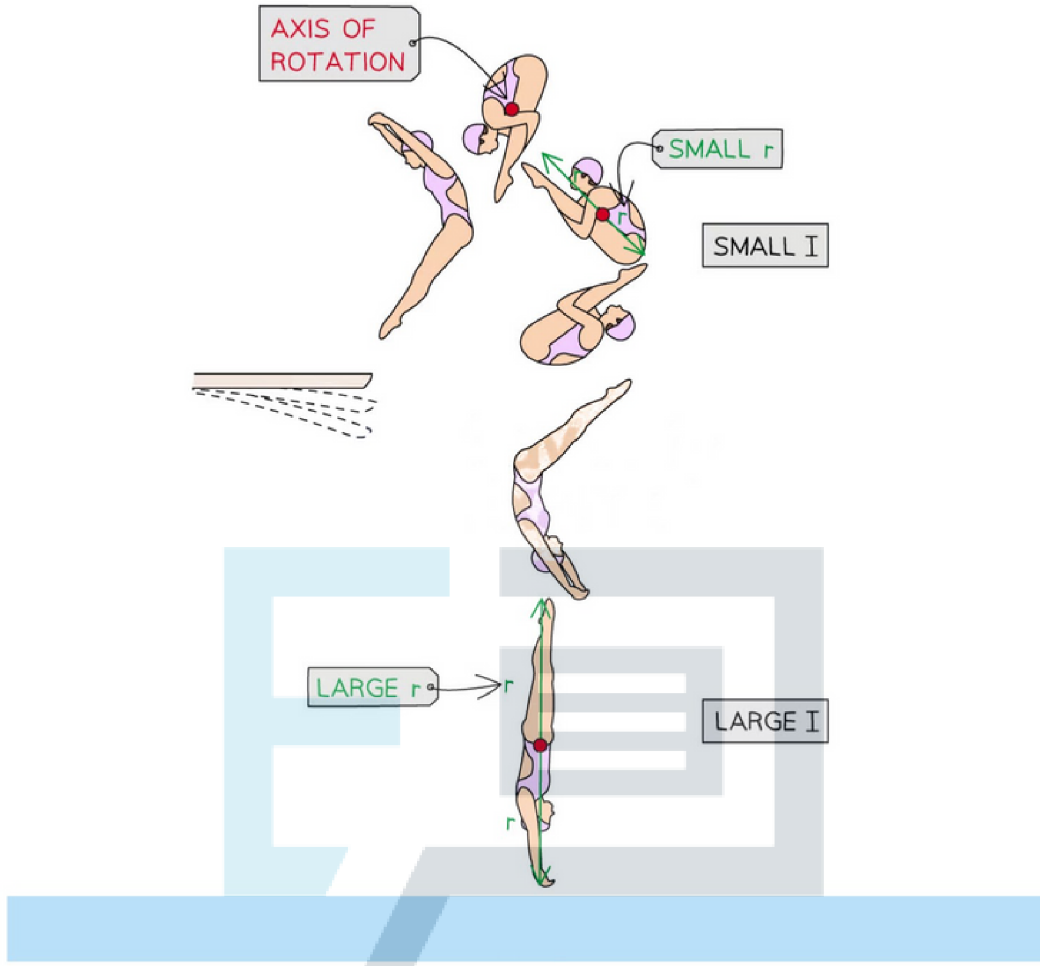
Moment of Inertia

- In linear motion, the resistance to a change of motion, i.e. linear acceleration, is known as **inertia**
  - The **larger** the mass an object has, the **greater** its inertia
- In rotational motion, the distribution of mass around an axis must be considered, using **moments of inertia**
  - This is the rotational equivalent of **mass**
- The moment of inertia of a rigid, extended body is defined as:

  The resistance to a change of rotational motion, depending on the distribution of mass around a chosen axis of rotation

- Moment of inertia is measured in kg m^2
- The moment of inertia of a body corresponds to how 'easy' or 'hard' it is to rotate, and this is dependent on many factors, including
  - The total **mass** (*m*)
  - How its **mass** is **distributed** about the axis of rotation (*r*)
    - For example, if a springboard diver jumps off a board and does a flip, they tuck their legs closer to their chest. This decreases their moment of inertia, as more of their mass is distributed over a smaller distance. This makes it easier for them to rotate

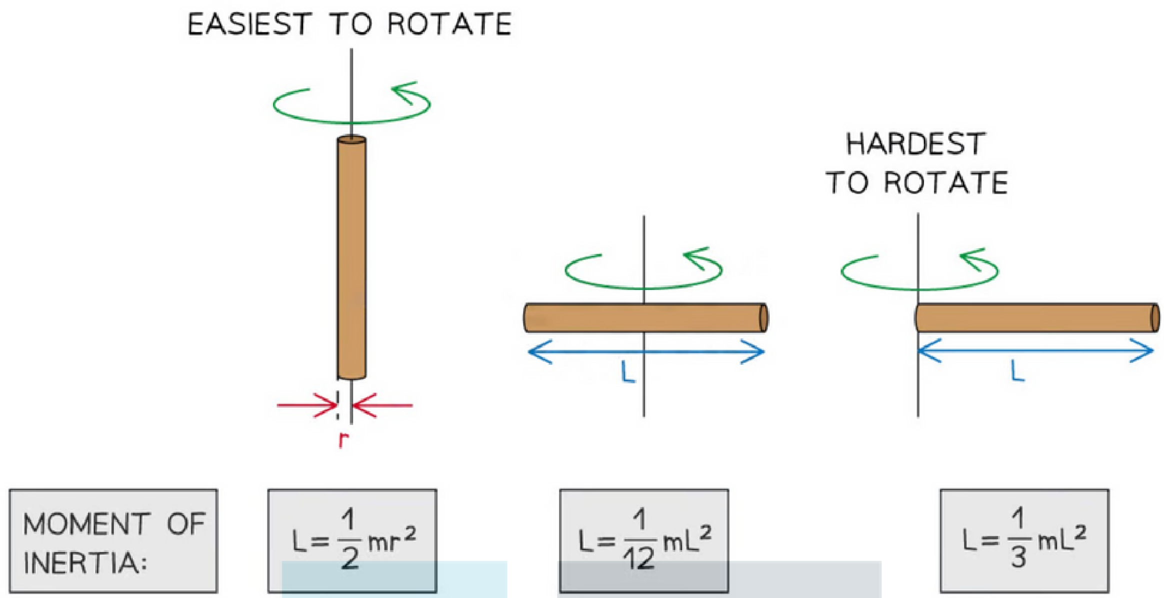
The change in the moment of inertia of a diver



The distance from the axis of rotation changes as the diver curls up and straightens out again

- This also means that the moment of inertia of a singular object can **change** depending on its orientation in relation to the chosen axis of rotation
- For example, the moment of inertia of a **thin rod** is different for each of the following orientations:
  - Rotation about its **vertical axis**
  - Rotation about its **centre of mass**
  - Rotation about **one end**

Different orientations of a thin rod have different moments of inertia



The moment of inertia of a body can change depending on its orientation relative to the axis of rotation

- These are just a few of the possible orientations of the axis of rotation for a thin rod
    - There is an **infinite** range of possible axes, and therefore an infinite possible set of values for the moments of inertia
    - This also applies to nearly all rigid, extended objects that could be considered

Calculating Moments of Inertia

- The moment of inertia I of a point mass is equal to

$$I = mr^2$$

- Where:
  - I = moment of inertia (kg m^2)
  - m = mass of the object (kg)
  - r = distance from its axis of rotation (m)

- The moment of inertia for an **extended** object about an axis is defined as the **summation** of the *mass* × *radius*^{2} for **all** the particles that make up the body

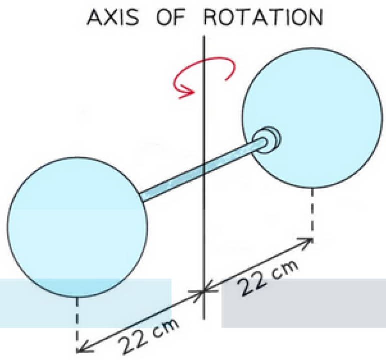
$$I = \sum mr^2$$

- This gives the total moment of inertia of the system

Worked Example

Two solid spheres form a dumbbell when attached to each end of a thin rod. The dumbbell rotates with the centre of mass of each sphere at a distance of 22 cm from the axis of rotation, as shown in the diagram.

The thin rod has a mass of 20 g. Each sphere has a radius of 4 cm and a mass of 750 g.



Moment of inertia of a thin rod about its centre = $\frac{1}{12}mL^2$

Moment of inertia of a solid sphere = $\frac{2}{5}mr^2$

Determine

(a)
the overall moment of inertia of the dumbbell arrangement
(b)
the ratio of the moment of inertia of the thin rod to the overall moment of inertia of the dumbbell arrangement

Answer:

(a)

- The overall moment of inertia of the dumbbell is the sum of all the moments of inertia in the arrangement

$$I = \sum mr^2 = 2 \times \left(\frac{2}{5}m_{sphere}r^2\right) + \frac{1}{12}m_{rod}L^2$$

- Where:
  - Mass of a sphere, m_{sphere} = 750 g = 0.75 kg
  - Distance from axis to each sphere, r = 22 cm = 0.22 m
  - Mass of the rod, m_{rod} = 20 g = 0.02 kg
  - Length of the rod, L = 2 × (22 – 4) = 36 cm = 0.36 m

$$I = 2 \times \left(\frac{2}{5} \times 0.75 \times 0.22^2\right) + \left(\frac{1}{12} \times 0.02 \times 0.36^2\right)$$

Moment of inertia of the dumbbell: $I = 0.029\ \text{kg m}^2$

(b)

- The moment of inertia of the thin rod is

$$I_{rod} = \frac{1}{12} \times 0.02 \times 0.36^2 = 2.16 \times 10^{-4}\ \text{kg m}^2$$

- Therefore, the ratio I_{rod} / I is

$$\frac{I_{rod}}{I} = \frac{2.16 \times 10^{-4}}{0.029} = 0.0071$$

- This means the rod contributes about 0.7% of the overall moment of inertia of the dumbbell

Exam Tip

You will never be expected to memorise the moments of inertia of different shapes, they will **always** be given in an exam question where required

Newton's Second Law for Rotation

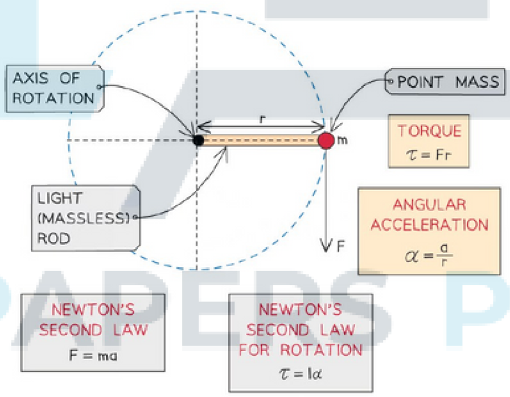
- In linear motion, the **force** required to give an object a certain **acceleration** depends on its **mass**

$$F = ma$$

- This is Newton's Second Law of **linear motion**, where:
  - F = force (N)
  - m = mass (kg)
  - a = linear acceleration (m s^{-2})
- In rotational motion, the **torque** required to give a rotating object a certain **angular acceleration** depends on its **moment of inertia**

$$\tau = I\alpha$$

- This is Newton's Second Law of **rotational motion**, where:
  - τ = torque (N m)
  - I = moment of inertia (kg m^2)
  - α = angular acceleration (rad s^{-2})



Newton's second law for rotating bodies is equivalent to Newton's second law for linear motion

- This equation comes from the fact that **torque** is the rotational equivalent of **force**:

Force: $F = ma$

Torque: $\tau = Fr$

- Where:
  - r = perpendicular distance from the axis of rotation (m)
- Combining these equations gives:

$$\tau = r(ma)$$

- The **moment of inertia** of a rotating body can be thought of as analogous to (the same as) **mass**
  - The inertia of a mass describes its ability to resist changes to **linear** motion, which is referring to **linear acceleration**
  - Similarly, the moment of inertia of a mass describes its ability to resist changes to **rotational** motion, which is referring to **angular acceleration**

$$\text{Angular acceleration: } \alpha = \frac{a}{r}$$

$$\text{Moment of inertia (point mass): } I = mr^2$$

- Using these equations with the equations for force and torque leads to:

$$\tau = r(mr\alpha)$$

$$\tau = (mr^2)\alpha$$

$$\tau = I\alpha$$

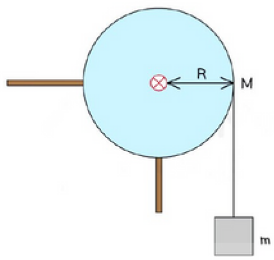
Comparison of linear and rotational variables in Newton's Second Law

Linear variable	Rotational variable
Force, F	Torque, τ
Mass, m	Moment of inertia, I
Acceleration, a	Angular acceleration, α
Newton's Second Law, $F \propto a$	Newton's Second Law, $\tau \propto \alpha$
$F = ma$	$\tau = I\alpha$

Worked Example

A block of mass *m* is attached to a string that is wrapped around a cylindrical pulley of mass *M* and radius *R*, as shown in the diagram.

The moment of inertia of the cylindrical pulley about its axis is $\frac{1}{2}MR^2$.

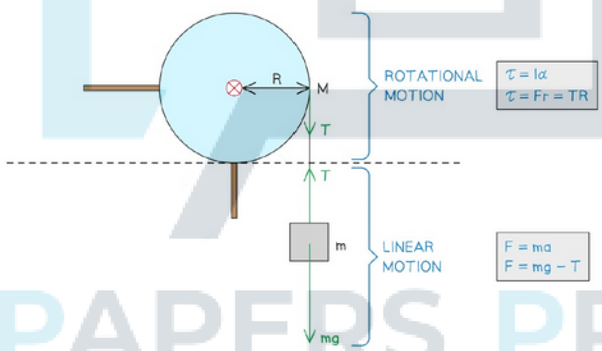


When the block is released, the pulley begins to turn as the block falls.

Write an expression for the acceleration of the block.

Answer:

Step 1: Identify the forces acting on the block



Step 2: Apply Newton's second law to the motion of the block

$$F = ma$$

$$mg - T = ma \text{ } \textbf{eq. (1)}$$

Step 3: Apply Newton's second law to the rotation of the pulley

$$\tau = I\alpha$$

$$TR = I\alpha$$

Step 4: Write the equation for the pulley in terms of acceleration a

- The angular acceleration α of the pulley is:

$$\alpha = \frac{a}{R}$$

- Substitute this into the previous equation:

$$TR = I\frac{a}{R}$$

- Substitute in the expression for the moment of inertia and simplify:

Moment of inertia of the cylinder: $I = \frac{1}{2}MR^2$

$$TR = \left(\frac{1}{2}MR^2\right)\frac{a}{R}$$

$$T = \left(\frac{1}{2}MR^2\right)\frac{a}{R^2} = \frac{1}{2}Ma$$

$$T = \frac{1}{2}Ma \text{ eq. (2)}$$

Step 5: Substitute eq. (2) into eq. (1) and rearrange for acceleration a

$$mg - \frac{1}{2}Ma = ma$$

$$mg = ma + \frac{1}{2}Ma = a\left(m + \frac{M}{2}\right)$$

Acceleration of the block: $a = \dfrac{mg}{m + \frac{M}{2}}$

11.1.6 Angular Momentum

Angular Momentum

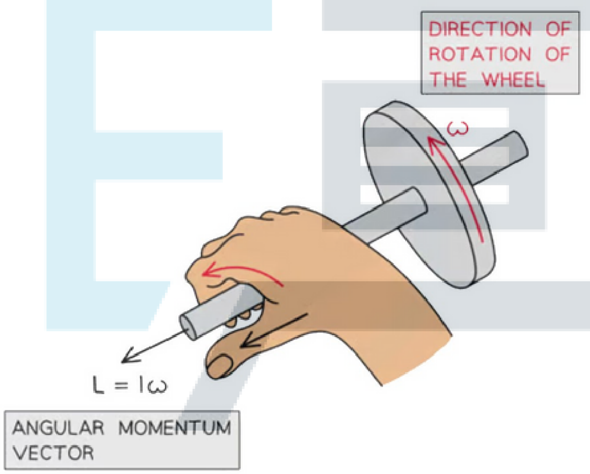
- Angular momentum is the **rotational equivalent** of linear momentum, which is defined by mass × velocity, or

$$p = mv$$

- Therefore, angular momentum L is defined by

$$L = I\omega$$

- Where:
  - L = angular momentum (kg m^2 rad s^{-1})
  - I = moment of inertia (kg m^2)
  - ω = angular velocity (rad s^{-1})



The angular momentum of a rotating wheel is the product of the moment of inertia and angular velocity of the wheel

Angular Momentum of a Point Mass

- The moment of inertia of a rotating point mass *m* which is a distance *r* from an axis of rotation is equal to

$$I = mr^2$$

- The angular velocity of the point mass is given by

$$\omega = \frac{v}{r}$$

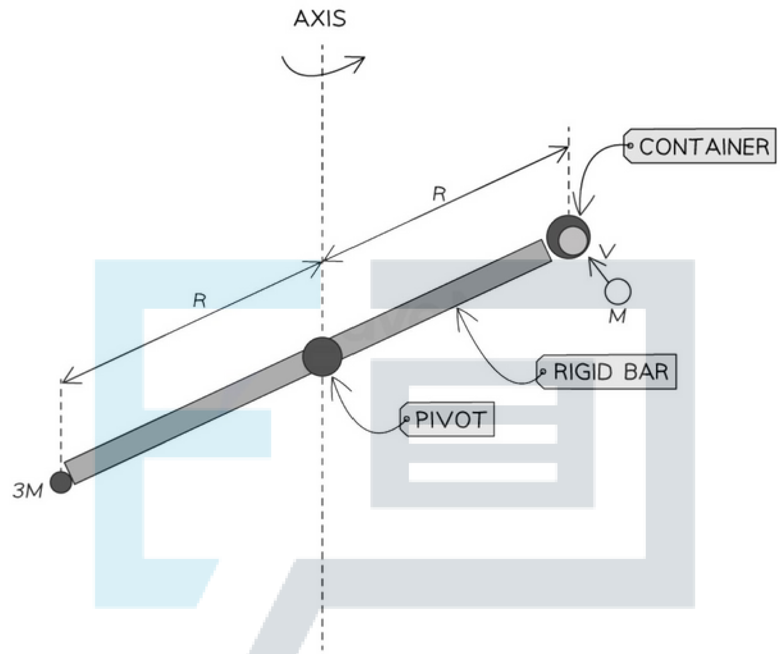
- Therefore, the angular momentum of the point mass is equal to

$$L = I\omega = (mr^2) \times \frac{v}{r} = mvr$$

Worked Example

A horizontal rigid bar is pivoted at its centre so that it is free to rotate. A point particle of mass $3M$ is attached at one end of the bar and a container is attached at the other end, both are at a distance of R from the central pivot.

A point particle of mass M moves with velocity v at right angles to the rod as shown in the diagram.



The particle collides with the container and stays within it as the system starts to rotate about the vertical axis with angular velocity ω.

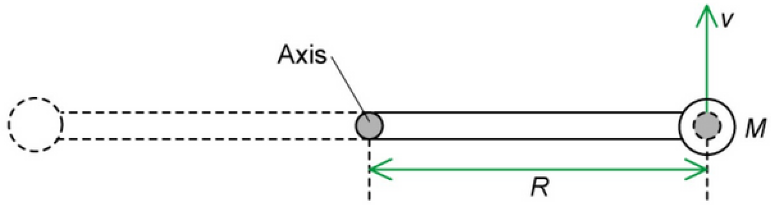
The mass of the rod and the container are negligible.

Write an expression for the angular momentum of the system about the vertical axis:

(a)
just before the collision, in terms of M, v and R
(b)
just after the collision, in terms of M, R and ω.

Answer:

(a) Just before the collision:



- Angular momentum, *L* is equal to:

$$L = I\omega$$

- The moment of inertia, *I* of a point particle is

$$I = mr^2$$

- Linear velocity, ω is related to angular velocity by

$$v = \omega r$$

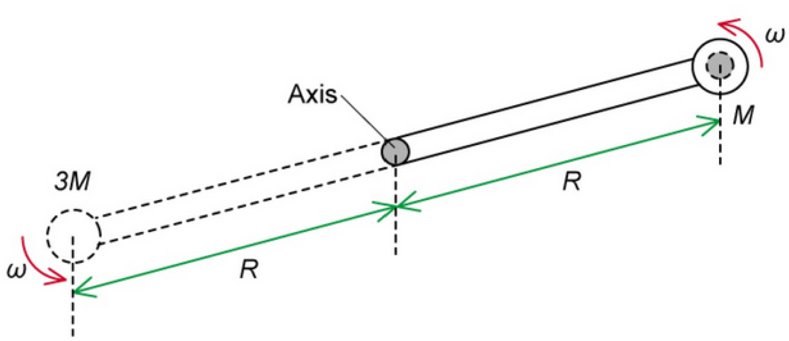
- The rod, container and 3*M* mass are all stationary before the collision, so we only need to consider the angular momentum of the point particle
- Where:
  - Mass of the particle, $m = M$
  - Distance of the particle from the axis, $r = R$
  - Angular velocity of the particle, $\omega = \frac{v}{R}$

- Therefore, the angular momentum of the system before the collision is:

$$L = I + \omega$$

$$L = (MR^2) \times \frac{v}{R} = MvR$$

(b) After the collision:



- The whole system rotates with an angular velocity of ω
- The rod and the container as massless, so we only need to consider the angular momentum of the two masses *M* and 3*M*

- Therefore, the angular momentum of the system after the collision is:

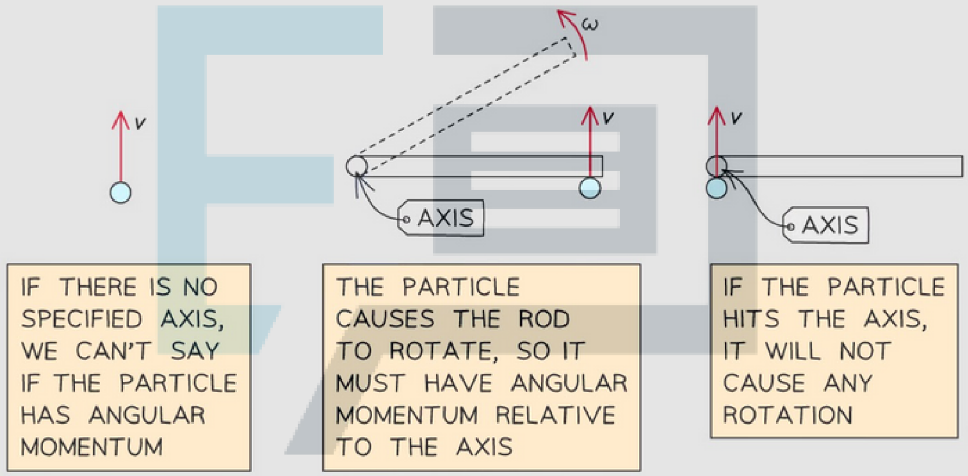
$$L = \Sigma(I\omega)$$

$$L = (MR^2)\omega + (3MR^2)\omega$$

$$L = 4MR^2\omega$$

Exam Tip

You should know that objects travelling in straight lines **can** have angular momentum - just make sure you understand that it all depends on the position of the object in relation to the axis of rotation being considered

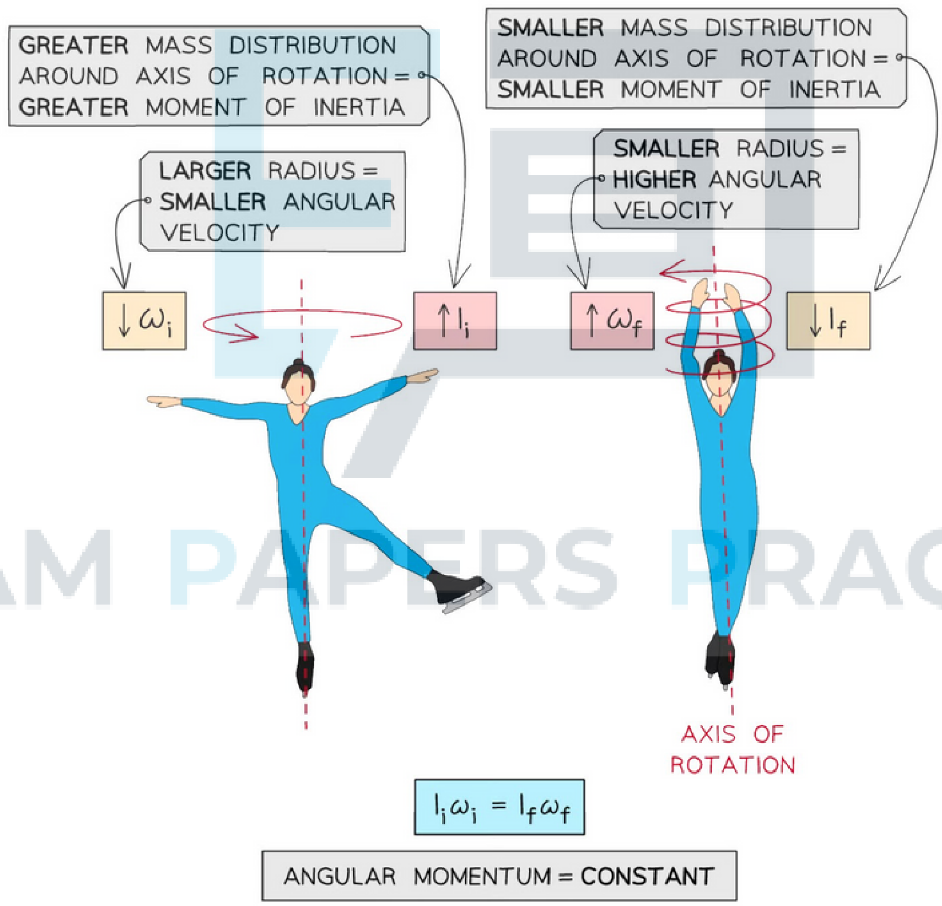


Conservation of Angular Momentum

- As with linear momentum, angular momentum is **always conserved**
- The principle of conservation of angular momentum states:

> **The angular momentum of a system always remains constant, unless a net torque is acting on the system**

- This conservation law has many real-world applications, for example
  - A person on a **spinning chair** spins faster while their arms and legs are contracted and slower while extended
  - Objects in **elliptical orbits** travel faster nearer the object they orbit and slower when further away
  - **Ice skaters** can change their rotational velocity by extending or contracting their arms
  - **Tornados** spin faster as their radius decreases



Ice skaters can change their moment of inertia by extending or contracting their arms and legs. Due to the conservation of angular momentum, this allows them to spin faster or slower

- Problems involving a **change in angular momentum** can be solved using the equation:

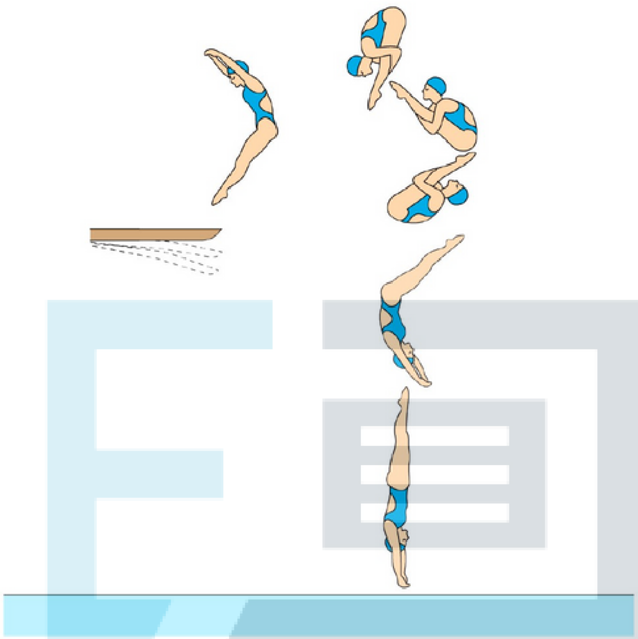
$$I_i\omega_i = I_f\omega_f$$

- Where:
  - I_i = initial moment of inertia (kg m^2)
  - ω_i = initial angular velocity (rad s^{-1})
  - I_f = final moment of inertia (kg m^2)
  - ω_f = final angular velocity (rad s^{-1})

Worked Example

The diagram shows the different positions of a diver between jumping off a springboard and entering the water.

During their fall, the diver pulls their arms and legs into a tight tuck position while in the air and straightens them before entering the water.



Which row correctly describes the changes to the diver's moment of inertia and angular velocity as they bring their limbs closer to their body?

	moment of inertia	angular velocity
A.	increases	increases
B.	decreases	increases
C.	increases	decreases
D.	decreases	decreases

Answer: B

- After the diver leaves the springboard, there is no longer a resultant torque acting on them
  - This means their angular momentum remains **constant** throughout the dive
- Due to the conservation of angular momentum:

$$I_i\omega_i = I_f\omega_f = constant$$

- When the diver tucks their arms and legs in closer to their body, they **decrease** their moment of inertia
  - This eliminates options **A** & **C**

- Therefore, to conserve angular momentum, when the diver's moment of inertia decreases, their angular velocity must **increase**

Worked Example

A spherical star of mass *M* and radius *R* rotates about its axis. The star explodes, ejecting mass in space radially and symmetrically. The remaining star is left with a mass of $\frac{1}{10}M$ and a radius of $\frac{1}{50}R$.

Calculate the ratio of the star's final angular velocity to its initial angular velocity.

The moment of inertia of a sphere is $\frac{2}{5}MR^2$

Answer:

- Before the star explodes:
  - Initial moment of inertia, $I_i = \frac{2}{5}MR^2$
  - Initial angular velocity = ω_i
- After the star explodes:
  - Final moment of inertia, $I_f = \frac{2}{5}\left(\frac{1}{10}M\right)\left(\frac{1}{50}R\right)^2$
  - Final angular velocity = ω_f
- From the conservation of angular momentum:

$$I_i\omega_i = I_f\omega_f$$

$$\left(\frac{2}{5}MR^2\right)\omega_i = \left(\frac{2}{5} \times \frac{1}{10}M \times \left(\frac{1}{50}\right)^2 R^2\right)\omega_f$$

$$\left(\cancel{\frac{2}{5}MR^2}\right)\omega_i = \left(\cancel{\frac{2}{5}MR^2}\right)\frac{1}{25\,000}\omega_f$$

- Therefore, the ratio of the star's final angular velocity to its initial angular velocity is:

$$\frac{\omega_f}{\omega_i} = 25\,000$$

11.1.7 Angular Impulse

Angular Impulse

- In linear motion, the **resultant force** on a body can be defined as the rate of change of linear momentum:

$$F = \frac{\Delta p}{\Delta t}$$

- This leads to the definition of linear impulse:

An average resultant force F acting for a time Δt produces a change in linear momentum Δp

$$\Delta p = F\Delta t = \Delta(mv)$$

- Similarly, the **resultant torque** on a body can be defined as the rate of change of angular momentum:

$$\tau = \frac{\Delta L}{\Delta t}$$

- Where:
  - τ = resultant torque on a body (N m)
  - ΔL = change in angular momentum (kg m^2 s^{-1})
  - Δt = time interval (s)
- This leads to the definition of angular impulse:

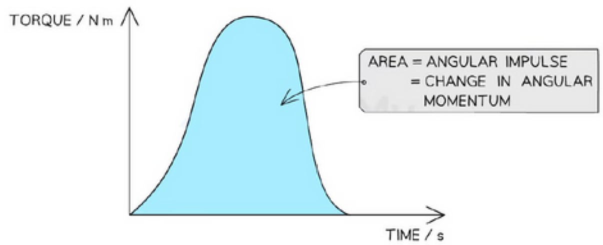
An average resultant torque τ acting for a time Δt produces a change in angular momentum ΔL

$$\Delta L = \tau\Delta t = \Delta(I\omega)$$

- Angular impulse is measured in kg m^2 s^{-1}, or N m s
- This equation requires the use of a **constant** resultant torque
  - If the resultant torque changes, then an **average** of the values must be used
- Angular impulse describes the effect of a torque acting over a time interval
  - This means a **small torque** acting over a **long time** has the same effect as a **large torque** acting over a **short time**

Angular Impulse on a Torque-Time Graph

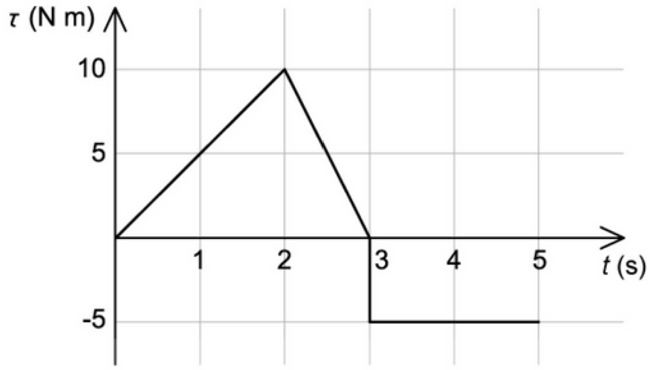
- The area under a torque-time graph is equal to the angular impulse or the change in angular momentum
  - This is because the area, angular impulse, is a fraction of the base × height, torque × time, $\Delta L = \tau\Delta t$
  - The fraction of torque × time depends upon the shape of the area under the graph



When the torque is not constant, the angular impulse is the area under a torque–time graph

Worked Example

The graph shows the variation of time t with the net torque τ on an object which has a moment of inertia of 6.0 kg m^2.



At t = 0, the object rotates with an angular velocity of 2.0 rad s^{-1} clockwise.

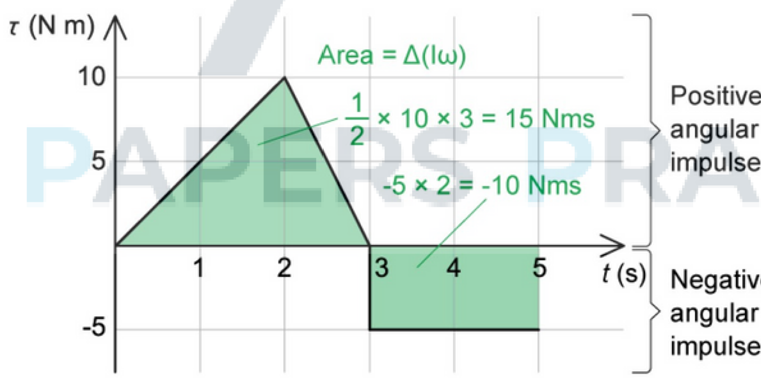
Determine the magnitude and direction of rotation of the angular velocity at t = 5 s.

In this question, take anticlockwise as the positive direction.

Answer:

- The area under a torque-time graph is equal to angular impulse, or the change in angular momentum

$$\Delta L = \tau \times \Delta t$$



- The area under the positive curve (triangle) = $\frac{1}{2} \times 10 \times 3 = 15 \text{ N m s}$
- The area under the negative curve (rectangle) = $-5 \times 2 = -10 \text{ N m s}$
- Therefore, the overall change in angular momentum is

$$\Delta L = 15 - 10 = 5 \text{ N m s}$$

- The change in angular momentum is equal to

$$\Delta L = \Delta(I\omega) = I(\omega_f - \omega_i)$$

- Where
  - Moment of inertia, $I = 6.0$ kg m^2
  - Initial angular velocity, $\omega_i = -2.0$ rad s^{-1} (clockwise is the negative direction)
- Therefore, when t = 5 s, the angular velocity is

$$6 \times (\omega_f - (-2)) = 5$$

$$6\omega_f + (2 \times 6) = 5$$

$$6\omega_f = 5 - 12$$

$$\omega_f = \frac{-7}{6} = -1.17$$

Anti-clockwise is positive, so $\omega_f = 1.17$ rad s^{-1} in the clockwise direction

Exam Tip

Many applications for angular impulse will be related to sports. These are similar to the linear impulse topic.

11.1.8 Rotational Work & Power

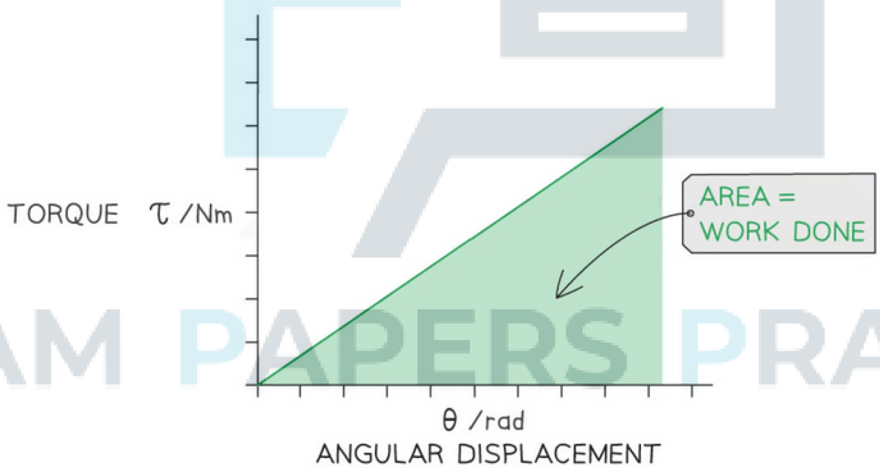
Work Done & Torque

Work Done by a Rotating Object

- Work has to be done on a rigid body when a torque turns in through an angle about an axis
  - For example, rotating cranes and fairground rides
- In systems with linear acceleration, work W is the product of the force and the distance moved
- Therefore, the work done for a **rotating** object is defined by the equation

$$W = \tau\theta$$

- Where:
  - W = work done (J)
  - τ = torque (N m)
  - θ = angular displacement (the angle turned through by the rotating object) (rads)
- Work can also be calculated by finding the **area** under a torque-angular displacement graph



The work done is the area under the torque-angular displacement graph

- This is analogous to the work done being the [area under a force-displacement graph]

Power Output of a Rotating Object

- Power is the **rate** of doing work, and is defined by

$$P = \frac{\Delta W}{\Delta t} = \frac{\Delta \tau\theta}{\Delta t} = \tau\frac{\Delta\theta}{\Delta t}$$

$$P = \tau\omega$$

- Where:

- P = power (W)
- ω = angular velocity (rad s^{-1})
- This equation is the angular version of the linear equation $P = Fv$

Exam Tip

Don't forget that θ is always in **radians** when you're doing conversions from revs s^{-1} or rev min^{-1}.

Frictional Torque

- In rotational mechanics, frictional forces produce a specific torque called **frictional torque**
  - This is the torque caused by the frictional force when two objects in contact move past each other
- Frictional torque can be defined as:

  The difference between the applied torque and the resulting net, or observed, torque

- This means that the **net** torque is the **sum** of the **applied** and **frictional** torque

Net torque = applied torque + frictional torque

- In **rotating machinery**, power has to be expended to overcome frictional torque
  - This is due to resistive forces within the machinery
- In most cases, frictional torque is minimised to reduce the kinetic energy losses transferred to heat and sound
- However, sometimes a frictional torque is applied, such as when using a screwdriver
  - When a screwdriver is gripped and turned, this **increases** its rotational kinetic energy
- The frictional force must always be **added** to the torque resulting from a force to get the **total** torque in the system
- Frictional torque is calculated using the same equations as torque

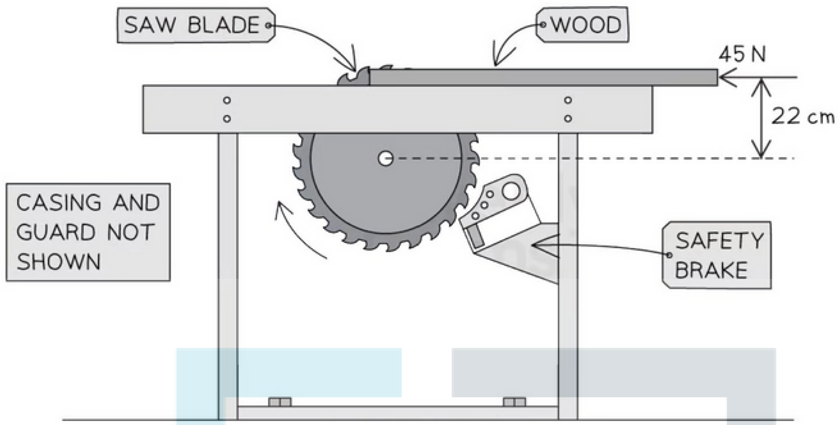
$$\tau = Fr = I\alpha$$

- The only difference is *F* is the **frictional force** instead of an externally applied force

Worked Example

The figure below shows a type of circular saw. The blade is driven by an electric motor and rotates at 3100 rev min^{-1} when cutting a piece of wood.

A constant frictional torque of 2.7 N m acts at the bearings of the motor and axle.



A horizontal force of 45 N is needed to push a piece of wood into the saw. The force acts on the blade at an effective radius of 22 cm.

Calculate the output power of the motor when the saw is cutting the wood.

Answer:

Step 1: Calculate the torque on the saw blade

$$\tau = Fr = 45 \times 0.22 = 9.9 \text{ N m}$$

Step 2: Calculate the total torque

Total torque = torque on the saw blade + frictional torque

Total torque = 9.9 + 2.7 = 12.6 N m

Step 3: Calculate the angular velocity

1 revolution = 2π radians

3100 rev min^{-1} = 3100 × 2π

$min^{-1} \rightarrow sec^{-1}$ = ÷ 60

$$3100 \text{ rev min}^{-1} \times \frac{2\pi}{60} = 324.63 \text{ rad s}^{-1}$$

Step 4: Calculate the output power

$$P = \tau\omega$$

$$P = 12.6 \times 324.63 = 4090.338 = 4100 \text{ W}$$

11.1.9 Rotational Kinetic Energy

Rotational Kinetic Energy

- A body moving with **linear velocity** has an associated linear kinetic energy given by

$$E_k = \frac{1}{2}mv^2$$

$$E_k = \frac{p^2}{2m}$$

- Similarly, a rotating body with **angular velocity** has an associated **rotational kinetic energy** given by

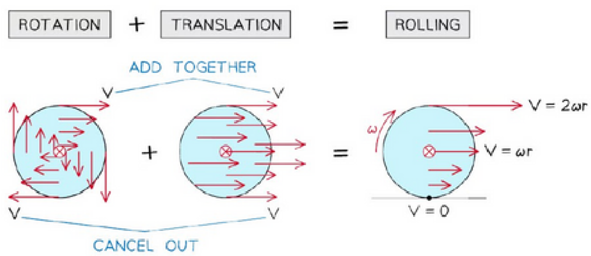
$$E_k = \frac{1}{2}I\omega^2$$

$$E_k = \frac{L^2}{2I}$$

- Where:
  - E_k = rotational kinetic energy (J)
  - I = moment of inertia (kg m^2)
  - ω = angular velocity (rad s^{-1})
  - L = angular momentum (kg m^2 s^{-1})

Rolling without slipping

- Circular objects, such as wheels, are made to move with **both** linear and rotational motion
  - For example, the wheels of a car, or bicycle **rotate** causing it to move **forward**
- Rolling motion without slipping is a combination of **rotating** and **sliding** (translational) motion
- When a disc **rotates**:
  - Each point on the disc has a **different** linear velocity depending on its distance from the centre $(v \propto r)$
  - The linear velocity is the **same** at all points on the **circumference**
- When a disc **slips**, or **slides**:
  - There is **not enough friction** present to allow the object to roll
  - Each point on the object has the **same** linear velocity
  - The angular velocity is **zero**
- So, when a disc **rolls without slipping**:
  - There is enough **friction** present to initiate rotational motion allowing the object to roll
  - The point in contact with the surface has a **velocity of zero**
  - The centre of mass has a velocity of $v = \omega r$
  - The top point has a velocity of $2v$ or $2\omega r$



Rolling motion is a combination of rotational and translational motion. The resultant velocity at the bottom is zero and the resultant velocity at the top is 2v

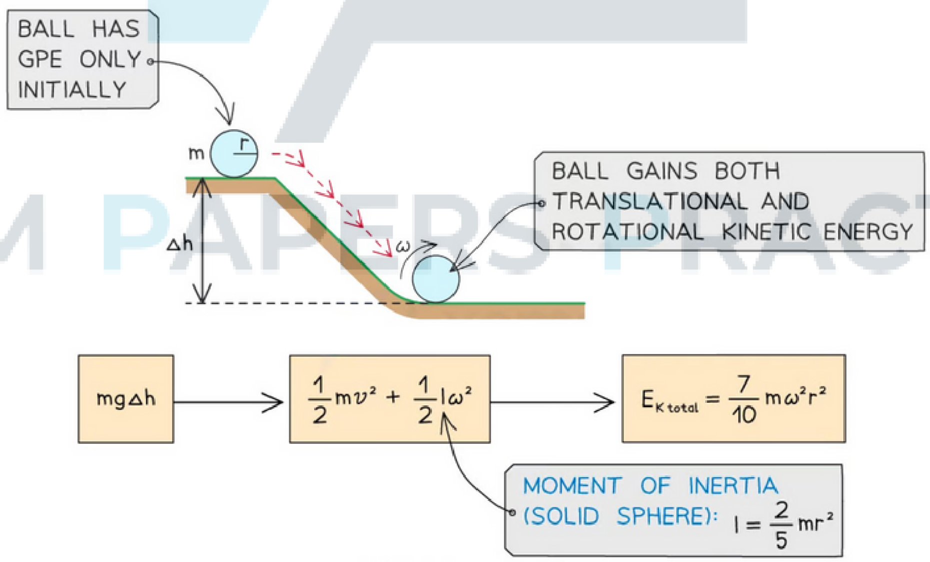
Rolling down a slope

- Another common scenario involving rotational and translational motion is an object (usually a ball or a disc) **rolling down a slope**
- At the **top** of the slope, a stationary object will have gravitational potential energy equal to

$$E_p = mg\Delta h$$

- As the object **rolls** down the slope, the gravitational potential energy will be transferred to **both** translational (linear) and rotational kinetic energy
- At the **bottom** of the slope, the **total kinetic energy** of the object will be equal to

$$E_{K\ total} = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$



The GPE store of the ball is transferred to the translational and rotational kinetic energy store as it rolls down the slope

- The linear or angular velocity can then be determined by
  - Equating E_p and $E_{K\ total}$
  - Using the equation for the moment of inertia of the object

- Using the relationship between linear and angular velocity $v = \omega r$

- For example, for a ball (a solid sphere) of mass *m* and radius *r*, its moment of inertia is

$$I = \frac{2}{5}mr^2$$

- Equating the equations for E_p and $E_{K\ total}$ and simplifying gives

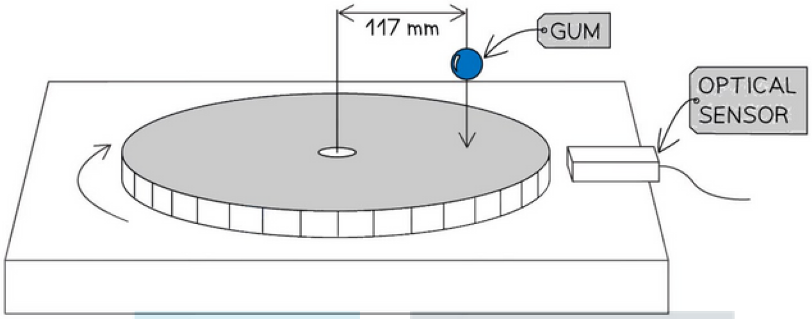
$$mg\Delta h = \frac{1}{2}m(\omega r)^2 + \frac{1}{2}\left(\frac{2}{5}mr^2\right)\omega^2$$

$$mg\Delta h = \frac{1}{2}m\omega^2 r^2 + \frac{1}{5}m\omega^2 r^2$$

$$mg\Delta h = \frac{7}{10}m\omega^2 r^2$$

Worked Example

A student carries out an experiment to determine the moment of inertia of a turntable. The diagram shows the turntable with a small lump of gum held above it. An optical sensor connected to a data recorder measures the angular speed of the turntable.



The turntable is made to rotate and then it rotates freely. The lump of gum is dropped from a small height above the turntable and sticks to it. The results from the experiment are as follows.

mass of gum = 9.0 g

radius at which gum sticks to the turntable = 117 mm

angular speed of turntable immediately before gum is dropped = 3.28 rad s^{-1}

angular speed of turntable immediately after gum is dropped = 3.11 rad s^{-1}

Calculate the change in rotational kinetic energy of the turntable and gum from the instant before the gum is dropped until immediately after it sticks to the turntable.

Answer:

Step 1: State the rotational kinetic energy equation before the gum is dropped

- ω_1 = 3.28 rad s^{-1} (before gum dropped)
- ω_2 = 3.11 rad s^{-1} (after gum dropped)

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Step 2: State the rotational kinetic energy equation after the gum is dropped

- The gum has its own inertia, which must be added to the total inertia

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Step 3: State the difference between the rotational energies equation

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Step 4: Calculate the moment of inertia of the turntable

- From the law of angular momentum, the angular momentum of the turntable before the gum is dropped is the same as the angular momentum after the gum is dropped
- *I* = angular momentum of the turntable
- *m* = mass of gum = 9.0 g
- *r* = radius at which gum sticks to the turntable = 117 mm

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Step 4: Substitute in the values

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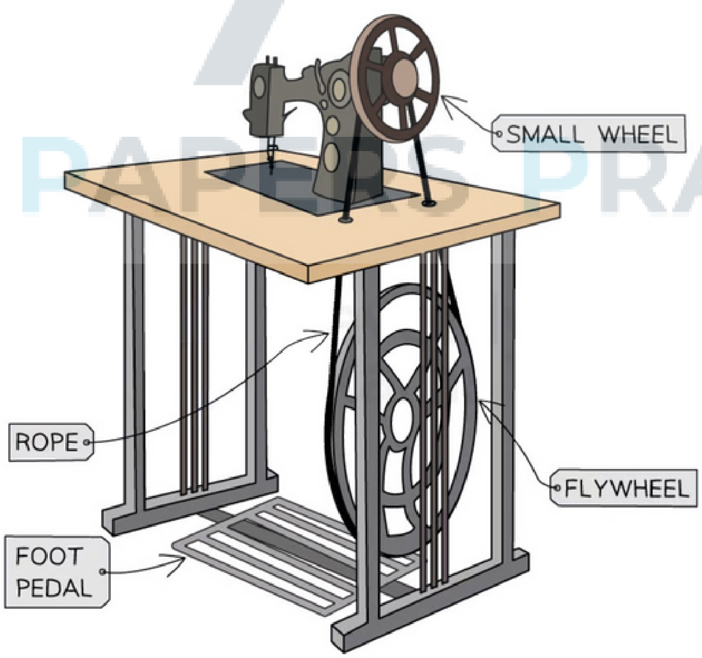
11.1.10 Flywheels in Machines

Flywheels in Machines

- Flywheels are used in machines to act as an energy reservoir, by storing and supplying energy when required
- They consist of a heavy metal disc or wheel that can rotate rapidly and so has a **large** moment of inertia
- This means it has:
  - a high **mass**
  - a large **radius**
- This means once they start spinning, it is difficult to make them stop
- An example is a treadle (pedal) sewing machine
  - This consists of a big flywheel, connected to a small wheel by a rope which drives the sewing machine
  - A pedal is pressed which causes the flywheel to rotate, and also rotates the smaller wheel which drives the machine
  - When the pedal is not pressed, the smaller wheel **will still rotate** for some time **due to the energy stored in the flywheel**
  - This is because the flywheel has **stored** the rotational energy, which it can now transfer for some time after there is no input. This is used extensively in machines to control energy transfers

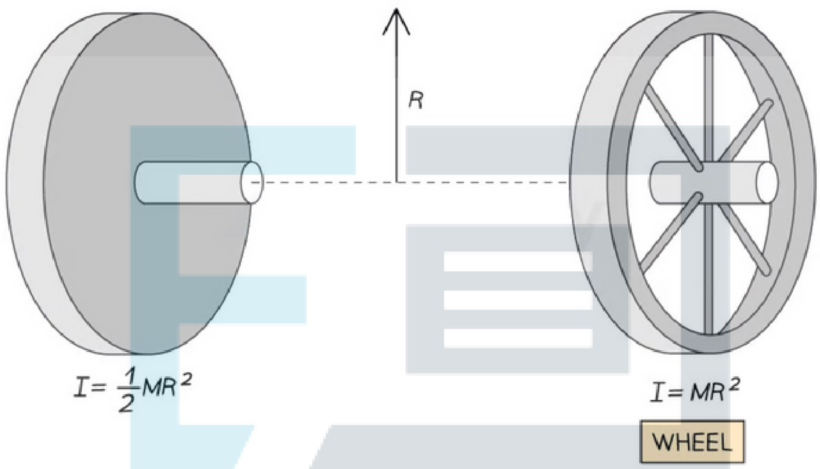
Application of a flywheel in a treadle sewing machine



A flywheel is used in a treadle sewing machine to create motion, even when the pedal is not pressed

- Flywheels are primarily used in **engines** in vehicles where they accumulate and **store energy**
- As it spins, its input torque is converted into rotational kinetic energy which is **stored** in the flywheel
  - This is a result of resisting the changes to rotation
  - The greater the moment of inertia of the flywheel, the greater the energy stored
  - This means a **hoop (wheel)-shaped** flywheel ($I = mr^2$) is preferred over a **disc-shaped** one ($I = \frac{1}{2}mr^2$)

Flywheel shapes: a uniform solid disc and a spoked wheel



Neglecting the mass of the spokes and axle, a disc-shaped flywheel has a smaller moment of inertia than a wheel-shaped one

- These flywheels were often fitted in large Victorian steam engines used in pumping stations and textile mills
  - They had a huge rim fitted with spokes
  - This gave a greater moment of inertia than if the same mass had been used to create a solid disc flywheel of the same diameter
- A flywheel transfers just enough power to a wheel to overcome frictional torque as it rotates
- When power is needed to the rest of the engine, the flywheel can reduce its speed and transfer some power

Exam Tip

Questions about flywheels involve calculating torque and moments of inertia, so make sure you're confident with these calculations. Flywheels are just one common application of torque and moment of inertia

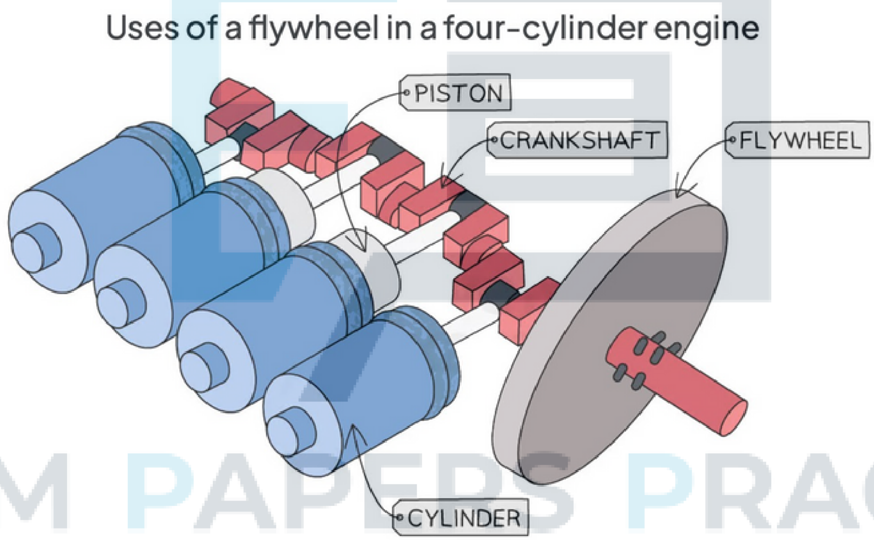
Flywheels for Production Processes

Uses of Flywheels

- Flywheels are used to:
  - **Smooth out fluctuations** in rotational speed / torque / power (such as in vehicles)
  - Store rotational kinetic energy

Smoothing Torque & Speed

- Power in an engine is not produced continuously, only in the 'power stroke' or 'combustion' part of an engine cycle, so it is released in bursts
  - This causes an engine to produce a torque that **fluctuates**
- The torque makes the flywheel rotate, moving a vehicle forwards
- If the torque is uneven, it will cause a jerky motion and unwanted vibrations will occur. This is a waste of energy and uncomfortable for the passengers
- The flywheel added will speed up or slow down over a period of time because of its inertia and as a result, the sharp fluctuations in torque are 'smoothed'



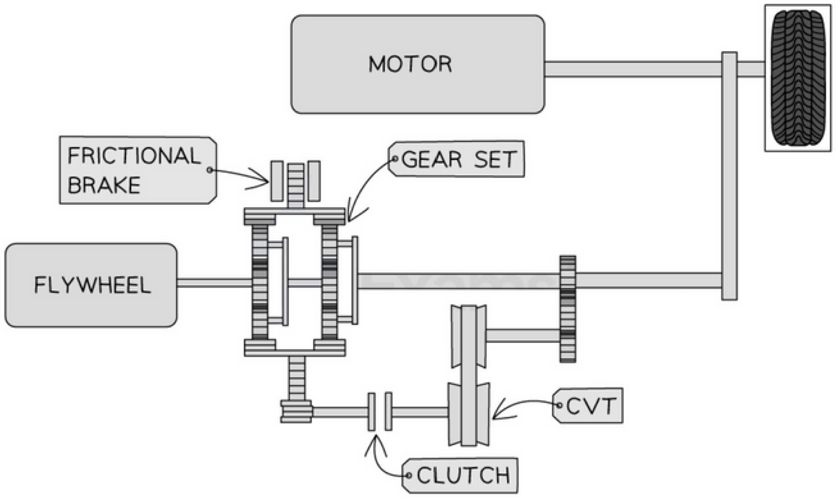
Flywheels smooth out the rotation of a crankshaft in a four-cylinder car

- The greater the moment of inertia of the flywheel, the smaller the fluctuation in speed

Regenerative Braking in Vehicles

- In conventional braking (say, on a bike), the kinetic energy store of the vehicle is transferred as waste through to the thermal energy store
- Instead, when regenerative brakes are applied, a flywheel is engaged and will 'charge up' by using the energy lost by braking
  - When the vehicle needs to accelerate later, the energy stored by the flywheel is used to do this
- These systems are sometimes called 'KERS' (kinetic energy recovery systems)

Diagram of a regenerative braking system



A regenerative braking system uses a flywheel which charges up from the energy lost by braking

Production Processes

- An electric motor in industrial machines can be used along with a flywheel
- The motor is used to charge up the flywheel, which can then transfer short burst of energy (such as needing to connect two materials together in a riveting machine)
- This prevents the motor from stalling, and a less powerful motor can be used

Factors Affecting the Energy Storage Capacity

- The **mass** of the flywheel
  - Since the moment of inertia, *I* is directly proportional to the mass, *m*, as mass increases the moment of inertia also increases
  - The rotational kinetic energy is directly proportional to the moment of inertia, so this also increases
- The **angular speed** of the flywheel
  - The rotational kinetic energy is proportional to the **square** of the angular speed
  - If the angular speed increases, the rotational kinetic energy stored also increases
- **Friction**
  - Although they are very efficient, flywheels can still lose some stored energy as friction and air resistance between the wheels and its bearings
  - The friction can be reduced by:
    - Lubricating bearings
    - Using bearings made of superconductors, so the flywheel can levitate and have no contact
    - Use the flywheel in a vacuum or sealed container to reduce air resistance
- The **shape** of the flywheel
  - For a solid **disc** of radius *R*, thickness *t*, mass *M* and density ρ

$$M = \rho V = \pi R^2 t \rho$$

  - The moment of inertia about the axis of rotation for a disc is

$$I = \frac{1}{2}MR^2 = \frac{1}{2}(\pi R^2 t\rho)R^2 = \frac{1}{2}(\pi t\rho)R^4$$

- The rotational kinetic energy is therefore

$$E_k = \frac{1}{2}I\omega^2$$

$$E_k = \frac{1}{2}\left(\frac{1}{2}(\pi t\rho)R^4\right)\omega^2$$

- Since t and ρ are constant

$$E_k \propto R^4\omega^2$$

- The rotational kinetic energy stored therefore depends on the moment of inertia, determined by the shape of the flywheel

Worked Example

A moving bus is powered by energy stored in a rapidly spinning flywheel. The bus travels downhill.

Suggest **two** advantages of keeping the flywheel connected to the driving wheels when the bus travels downhill.

Answer

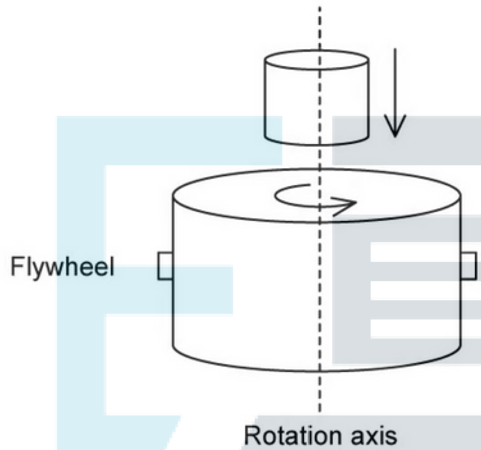
- The energy that would be otherwise dissipated in the brakes is now fed back to the flywheel
- The flywheel stores this energy and it will be used later when the bus is accelerating again

Worked Example

A flywheel of mass *M* and radius *R* rotates at a constant angular velocity ω about an axis through its centre. The rotational kinetic energy of the flywheel is E_K.

The moment of inertia of the flywheel is $\frac{1}{2}MR^2$.

A second flywheel of mass $\frac{1}{2}M$ and radius $\frac{1}{2}R$ is placed on top of the first flywheel. The new angular velocity of the combined flywheels is $\frac{2}{3}\omega$.



What is the new rotational kinetic energy of the combined flywheels?

A. $\frac{E_K}{2}$ **B.** $\frac{E_K}{4}$ **C.** $\frac{E_K}{8}$ **D.** $\frac{E_K}{24}$

Answer: **A**

- The kinetic energy of the first flywheel is

$$E_K = \frac{1}{2}I\omega^2 = \frac{1}{2} \times \left(\frac{1}{2}MR^2\right) \times \omega^2$$

$$E_K = \frac{1}{4}MR^2\omega^2$$

- The combined flywheels have a total moment of inertia of

$$I_{new} = I_1 + I_2$$

$$I_{new} = \frac{1}{2}MR^2 + \frac{1}{2}\left(\frac{1}{2}M\right)\left(\frac{1}{2}R\right)^2$$

$$I_{new} = \frac{9}{16}MR^2$$

- The kinetic energy of the combined flywheels is

$$E_{K\,new} = \frac{1}{2}I_{new}\omega_{new}^{\ 2} = \frac{1}{2} \times \left(\frac{9}{16}MR^2\right) \times \left(\frac{2}{3}\omega\right)^2$$

$$E_{K\,new} = \frac{1}{2} \times \left(\frac{1}{4}MR^2\omega^2\right) = \frac{1}{2}E_K$$

Exam Tip

A question might ask about the **function** of a flywheel, or an **application**. These are two different things.

The function is **why** we use a flywheel - this is to store rotational kinetic energy. An application might be degenerative braking