DP IB Maths: AA HL

1.9 Further Complex Numbers

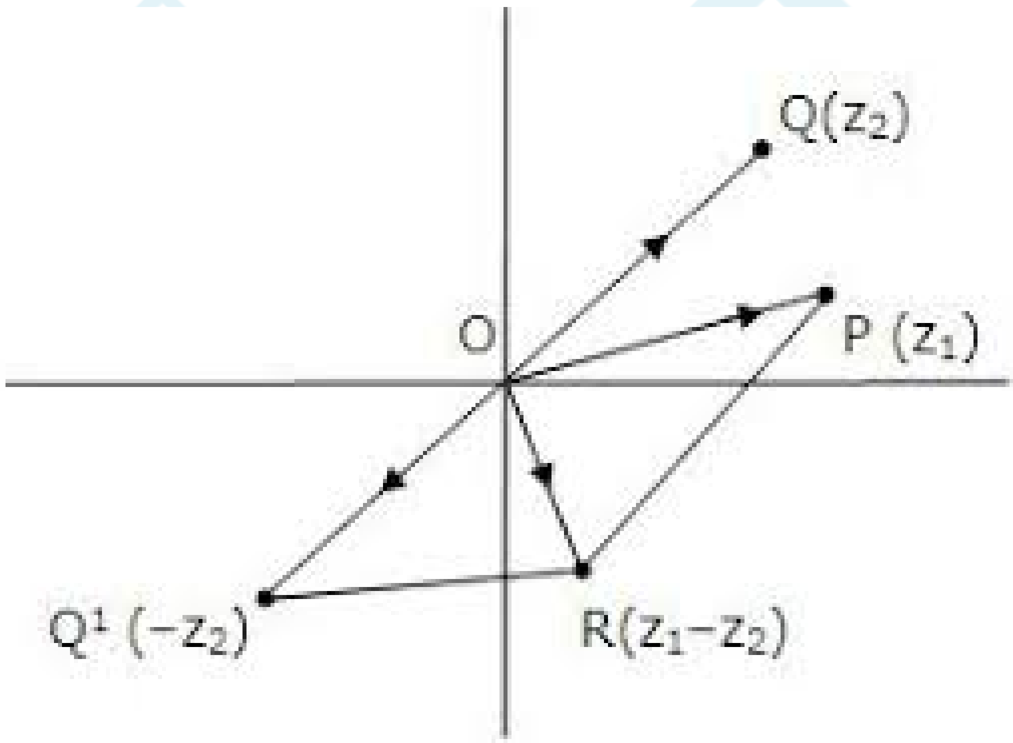
Contents

1.9.1 Geometry of Complex Numbers

Geometry of Complex Addit ion & Subtraction

What does addition look like on an Argand diagram?

- In Cartesian form two complex numbers are added by adding the real and imaginary parts
- When plotted on an Argand diagram the complex number $z_1 + z_2$ is the longer diagonal of the parallelogram with vertices at the origin, z_1, z_2 and $z_1 + z_2$



What does subtraction look like on an Argand diagram?

- In Cartesian form the difference of two complex numbers is found by subtracting the real and imaginary parts
- When plotted on an Argand diagram the complex number $z_1 - z_2$ is the shorter diagonal of the parallelogram with vertices at the origin, z_1, $-z_2$ and $z_1 - z_2$

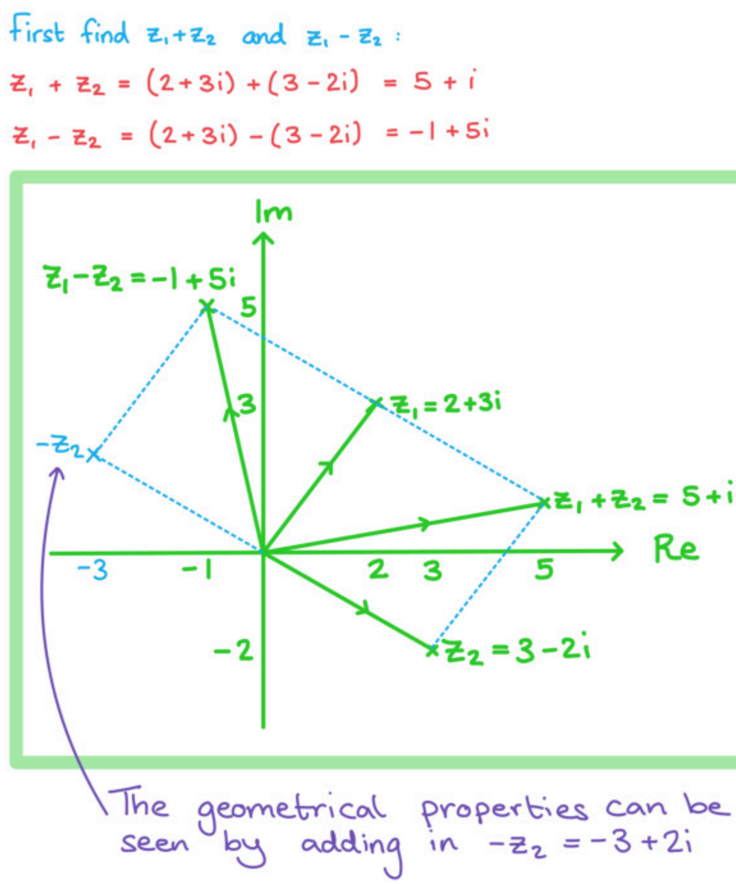
What are the geometrical representations of complex addition and subtraction?

- Let w be a given complex number with **real part a** and **imaginary part b**
  - $w = a + b\text{i}$
- Let z be any complex number represented on an Argand diagram
- **Adding w to z** results in z being:
  - **Translated** by vector $\begin{pmatrix} a \\ b \end{pmatrix}$
- **Subtracting w from z** results in z being:
  - **Translated** by vector $\begin{pmatrix} -a \\ -b \end{pmatrix}$

Worked example

Consider the complex numbers $z_1 = 2 + 3i$ and $z_2 = 3 - 2i$.

On an Argand diagram represent the complex numbers $z_1, z_2, z_1 + z_2$ and $z_1 - z_2$.



Geometry of Complex Multiplication & Division

What do multiplication and division look like on an Argand diagram?

- The geometrical effect of multiplying a complex number by a real number, a, will be an enlargement of the vector by scale factor a
  - For positive values of a the direction of the vector will not change but the distance of the point from the origin will increase by scale factor a
  - For negative values of a the direction of the vector will change and the distance of the point from the origin will increase by scale factor a
- The geometrical effect of dividing a complex number by a real number, a, will be an enlargement of the vector by scale factor $1/a$
  - For positive values of a the direction of the vector will not change but the distance of the point from the origin will increase by scale factor $1/a$
  - For negative values of a the direction of the vector will change and the distance of the point from the origin will increase by scale factor $1/a$
- The geometrical effect of multiplying a complex number by i will be a rotation of the vector 90° counter-clockwise
  - $\mathrm{i}(x + y\mathrm{i}) = -y + x\mathrm{i}$
- The geometrical effect of multiplying a complex number by an imaginary number, ai, will be a rotation 90° counter-clockwise and an enlargement by scale factor a
  - $a\mathrm{i}(x + y\mathrm{i}) = -ay + ax\mathrm{i}$
- The geometrical effect of multiplying or dividing a complex number by a complex number will be an enlargement and a rotation
  - The direction of the vector will change
    - The angle of rotation is the **argument**
  - The distance of the point from the origin will change
    - The scale factor is the **modulus**

What does complex conjugation look like on an Argand diagram?

- The geometrical effect of plotting a **complex conjugate** on an Argand diagram is a reflection in the real axis
  - The **real** part of the complex number will stay the same and the **imaginary** part will change sign

Worked example

Consider the complex number $z = 2 - i$.

On an Argand diagram represent the complex numbers z, 3z, iz, z^* and zz^*.

First find $3z$, iz and z^*

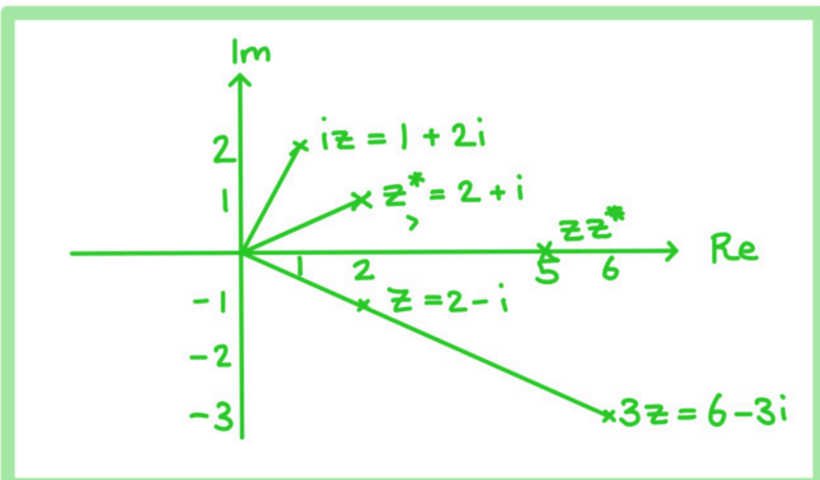
$z = 2 - i$

$3z = 3(2 - i) = 6 - 3i$

$iz = i(2 - i) = 2i - i^2 = 2i - (-1) = 1 + 2i$

$z^* = 2 + i$

$zz^* = (2 - i)(2 + i) = 4 - i^2 = 4 - (-1) = 5$



1.9.2 Forms of Complex Numbers

Modulus-Argument (Polar) Form

How do I write a complex number in modulus-argument (polar) form?

- The **Cartesian form** of a complex number, $z = x + \mathrm{i}y$, is written in terms of its real part, x, and its imaginary part, y
- If we let $r = |z|$ and $\theta = \arg z$, then it is possible to write a complex number in terms of its modulus, r, and its argument, θ, called the **modulus-argument (polar) form**, given by...
  - $z = r(\cos\theta + \mathrm{i}\sin\theta)$
  - This is often written as $z = r\,\mathrm{cis}\,\theta$
  - This is given in the formula book under Modulus-argument (polar) form and exponential (Euler) form
- It is usual to give arguments in the range $-\pi < \theta \le \pi$ or $0 \le \theta < 2\pi$
  - Negative arguments should be shown clearly
  - e.g. $z = 2\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right) = 2\,\mathrm{cis}\left(-\frac{\pi}{3}\right)$
    - without simplifying $\cos\left(-\frac{\pi}{3}\right)$ to either $\cos\left(\frac{\pi}{3}\right)$ or $\frac{1}{2}$
- The **complex conjugate** of $r\,\mathrm{cis}\,\theta$ is $r\,\mathrm{cis}\,(-\theta)$
- If a complex number is given in the form $z = r(\cos\theta - \mathrm{i}\sin\theta)$, then it is not in modulus-argument (polar) form due to the minus sign
  - It can be converted by considering transformations of trigonometric functions
    - $-\sin\theta = \sin(-\theta)$ and $\cos\theta = \cos(-\theta)$
  - So $z = r(\cos\theta - \mathrm{i}\sin\theta) = z = r(\cos(-\theta) + \mathrm{i}\sin(-\theta)) = r\,\mathrm{cis}\,(-\theta)$
- To convert from modulus-argument (polar) form back to Cartesian form, evaluate the real and imaginary parts
  - E.g. $z = 2\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$ becomes $z = 2\left(\frac{1}{2} + \mathrm{i}\left(-\frac{\sqrt{3}}{2}\right)\right) = 1 - \sqrt{3}\,\mathrm{i}$

How do I multiply complex numbers in modulus-argument (polar) form?

- The main benefit of writing complex numbers in modulus-argument (polar) form is that they multiply and divide very easily
- To **multiply** two complex numbers in modulus-argument (polar) form we **multiply their moduli** and **add their arguments**
  - $|z_1 z_2| = |z_1||z_2|$
  - $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- So if $z_1 = r_1\,\mathrm{cis}\,(\theta_1)$ and $z_2 = r_2\,\mathrm{cis}\,(\theta_2)$
  - $z_1 z_2 = r_1 r_2\,\mathrm{cis}\,(\theta_1 + \theta_2)$

- Sometimes the new argument, $\theta_1 + \theta_2$, does not lie in the range $-\pi < \theta \leq \pi$ (or $0 \leq \theta < 2\pi$ if this is being used)
  - An out-of-range argument can be adjusted by either **adding or subtracting** 2π
  - E.g. If $\theta_1 = \frac{2\pi}{3}$ and $\theta_2 = \frac{\pi}{2}$ then $\theta_1 + \theta_2 = \frac{7\pi}{6}$
  - This is currently not in the range $-\pi < \theta \leq \pi$
  - Subtracting 2π from $\frac{7\pi}{6}$ to give $-\frac{5\pi}{6}$, a new argument is formed
    - This lies in the correct range and represents the same angle on an Argand diagram
- The rules of **multiplying the moduli** and **adding the arguments** can also be applied when...
  - ...multiplying three complex numbers together, $z_1 z_2 z_3$, or more
  - ...finding powers of a complex number (e.g. z^2 can be written as zz)
- The rules for multiplication can be proved algebraically by multiplying $z_1 = r_1 \text{ cis } (\theta_1)$ by $z_2 = r_2 \text{ cis } (\theta_2)$, expanding the brackets and using compound angle formulae

How do I divide complex numbers in modulus-argument (polar) form?

- To **divide** two complex numbers in modulus-argument (polar) form, we **divide their moduli** and **subtract their arguments**
  - $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
  - $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$
- So if $z_1 = r_1 \text{ cis } (\theta_1)$ and $z_2 = r_2 \text{ cis } (\theta_2)$ then
  - $\frac{z_1}{z_2} = \frac{r_1}{r_2} \text{cis}\left(\theta_1 - \theta_2\right)$
- Sometimes the new argument, $\theta_1 - \theta_2$, can lie out of the range $-\pi < \theta \leq \pi$ (or the range $0 < \theta \leq 2\pi$ if this is being used)
  - You can **add or subtract** 2π to bring out-of-range arguments back in range
- The rules for division can be proved algebraically by dividing $z_1 = r_1 \text{ cis } (\theta_1)$ by $z_2 = r_2 \text{ cis } (\theta_2)$ using **complex division** and the compound angle formulae

Worked example

Let $z_1 = 4\sqrt{2}\ \text{cis}\ \frac{3\pi}{4}$ and $z_2 = \sqrt{8}\left(\cos\left(\frac{\pi}{2}\right) - \text{i}\sin\left(\frac{\pi}{2}\right)\right)$

a) Find $z_1 z_2$, giving your answer in the form $r(\cos\theta + \text{i}\sin\theta)$ where $0 \le \theta < 2\pi$

$z_1 = 4\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)$, $z_2 = \sqrt{8}\left(\cos\left(-\frac{\pi}{2}\right) + \text{i}\sin\left(-\frac{\pi}{2}\right)\right) = 2\sqrt{2}\ \text{cis}\left(-\frac{\pi}{2}\right)$

For $z_1 z_2$, multiply the moduli and add the arguments.

$z_1 z_2 = \left(4\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)\right)\left(2\sqrt{2}\ \text{cis}\left(-\frac{\pi}{2}\right)\right)$

$= (4\sqrt{2})(2\sqrt{2})\ \text{cis}\left(\frac{3\pi}{4} + \left(-\frac{\pi}{2}\right)\right)$

$= 16\ \text{cis}\left(\frac{\pi}{4}\right)$

$z_1 z_2 = 16\left(\cos\frac{\pi}{4} + \text{i}\sin\frac{\pi}{4}\right)$

b) Find $\frac{z_1}{z_2}$, giving your answer in the form $r(\cos\theta + \text{i}\sin\theta)$ where $-\pi \le \theta < \pi$

For $\frac{z_1}{z_2}$, divide the moduli and subtract the arguments

$\frac{z_1}{z_2} = \frac{4\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)}{2\sqrt{2}\ \text{cis}\left(-\frac{\pi}{2}\right)} = \frac{4\sqrt{2}}{2\sqrt{2}}\ \text{cis}\left(\frac{3\pi}{4} - \left(-\frac{\pi}{2}\right)\right)$

$= 2\ \text{cis}\left(\frac{5\pi}{4}\right)$

$= 2\ \text{cis}\left(\frac{5\pi}{4} - 2\pi\right)$

$\frac{5\pi}{4}$ is not in the range $-\pi \le \theta \le \pi$ so subtract 2π to bring it into range

$\frac{z_1}{z_2} = 2\left(\cos\left(\frac{-3\pi}{4}\right) + \text{i}\sin\left(\frac{-3\pi}{4}\right)\right)$

Exponential (Euler's) Form

How do we write a complex number in Euler's (exponential) form?

- A complex number can be written in Euler's form as $z = r\mathrm{e}^{\mathrm{i}\theta}$
  - This relates to the modulus-argument (polar) form as $z = r\mathrm{e}^{\mathrm{i}\theta} = r\ \mathrm{cis}\ \theta$
  - This shows a clear link between exponential functions and trigonometric functions
  - This is given in the formula booklet under 'Modulus-argument (polar) form and exponential (Euler) form'
- The argument is normally given in the range $0 \le \theta < 2\pi$
  - However in exponential form other arguments can be used and the same convention of adding or subtracting 2π can be applied

How do we multiply and divide complex numbers in Euler's form?

- Euler's form allows for quick and easy multiplication and division of complex numbers
- If $z_1 = r_1\mathrm{e}^{\mathrm{i}\theta_1}$ and $z_2 = r_2\mathrm{e}^{\mathrm{i}\theta_2}$ then
  - $z_1 \times z_2 = r_1 r_2 \mathrm{e}^{\mathrm{i}(\theta_1 + \theta_2)}$
    - Multiply the moduli and add the arguments
  - $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\mathrm{e}^{\mathrm{i}(\theta_1 - \theta_2)}$
    - Divide the moduli and subtract the arguments
- Using these rules makes multiplying and dividing more than two complex numbers much easier than in Cartesian form
- When a complex number is written in Euler's form it is easy to raise that complex number to a power
  - If $z = r\mathrm{e}^{\mathrm{i}\theta}$, $z^2 = r^2\mathrm{e}^{2\mathrm{i}\theta}$ and $z^n = r^n\mathrm{e}^{n\mathrm{i}\theta}$

What are some common numbers in exponential form?

- As $\cos(2\pi) = 1$ and $\sin(2\pi) = 0$ you can write:
  - $1 = \mathrm{e}^{2\pi\mathrm{i}}$
- Using the same idea you can write:
  - $1 = \mathrm{e}^{0} = \mathrm{e}^{2\pi\mathrm{i}} = \mathrm{e}^{4\pi\mathrm{i}} = \mathrm{e}^{6\pi\mathrm{i}} = \mathrm{e}^{2k\pi\mathrm{i}}$
  - where k is any integer
- As $\cos(\pi) = -1$ and $\sin(\pi) = 0$ you can write:
  - $\mathrm{e}^{\pi\mathrm{i}} = -1$
  - Or more commonly written as $\mathrm{e}^{\mathrm{i}\pi} + 1 = 0$
    - This is known as Euler's identity and is considered by some mathematicians as the most beautiful equation

- As $\cos\left(\frac{\pi}{2}\right)=0$ and $\sin\left(\frac{\pi}{2}\right)=1$ you can write:
  - $\mathrm{i}=\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}$

Worked example

Consider the complex number $z=2\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}$. Calculate z^2 giving your answer in the form $r\mathrm{e}^{\mathrm{i}\theta}$.

$$z^2=\left(2e^{\frac{\pi}{3}i}\right)^2=\left(2e^{\frac{\pi}{3}i}\right)\left(2e^{\frac{\pi}{3}i}\right)=4e^{2\left(\frac{\pi}{3}i\right)}$$

multiply the moduli
add the arguments

$$z^2=4e^{\frac{2\pi}{3}i}$$

Conversion of Forms

Converting from Cartesian form to modulus-argument (polar) form or exponential (Euler's) form

- To convert from Cartesian form to modulus-argument (polar) form or exponential (Euler) form use
  - $r = |z| = \sqrt{x^2 + y^2}$
- and
  - $\theta = \arg z$

Converting from modulus-argument (polar) form or exponential (Euler's) form to Cartesian form

- To convert from modulus-argument (polar) form to Cartesian form
  - You may need to use your knowledge of trig exact values
  - $a = r\cos\theta$ and $b = r\sin\theta$
  - Write $z = r(\cos\theta + i\sin\theta)$ as $z = r\cos\theta + (r\sin\theta)i$
  - Find the values of the trigonometric ratios $r\sin\theta$ and $r\cos\theta$
  - Rewrite as $z = a + bi$ where
- To convert from exponential (Euler's) form to Cartesian form first rewrite $z = re^{i\theta}$ in the form $z = r\cos\theta + (r\sin\theta)i$ and then follow the steps above

Converting between complex number forms using your GDC

- Your GDC may also be able to convert complex numbers between the various forms
  - TI calculators, for example, have 'Convert to Polar' and 'Convert to Rectangular' (i.e. Cartesian) as options in the 'Complex Number Tools' menu
  - Make sure you are familiar with your GDC and what it can (and cannot) do with complex numbers

Worked example

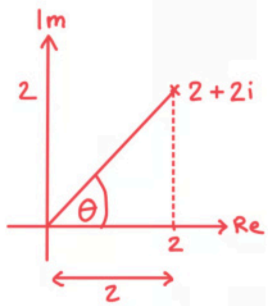
Two complex numbers are given by $z_1 = 2 + 2\mathrm{i}$ and $z_2 = 3\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}$.

a) Write z_1 in the form $r\mathrm{e}^{\mathrm{i}\theta}$.

$z_1 = 2 + 2i$

Find the modulus: $|z_1| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

Draw a sketch to help find the argument:



$\theta = \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1)$

$= \frac{\pi}{4}$

$r = 2\sqrt{2}, \quad \theta = \frac{\pi}{4}$

$$z_1 = 2\sqrt{2}\,e^{\frac{\pi}{4}i}$$

b) Write z_2 in the form $r(\cos\theta + \mathrm{i}\sin\theta)$ and then convert it to Cartesian form.

$z_2 = 3e^{\frac{2\pi}{3}i} = 3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= 3\left(-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right)$

$$z_2 = \frac{3}{2}(-1 + \sqrt{3}i)$$

1.9.3 Complex Roots of Polynomials

Complex Roots of Quadratics

What are complex roots?

- A quadratic equation can either have two real roots (zeros), a repeated real root or no real roots
  - This depends on the location of the graph of the quadratic with respect to the x-axis
- If a quadratic equation has no real roots we would previously have stated that it has **no real solutions**
  - The quadratic equation will have a **negative discriminant**
  - This means taking the square root of a negative number
- Complex numbers provide solutions for quadratic equations that have **no real roots**

How do we solve a quadratic equation when it has complex roots?

- If a quadratic equation takes the form $ax^2 + bx + c = 0$ it can be solved by either using the quadratic formula or completing the square
- If a quadratic equation takes the form $ax^2 + b = 0$ it can be solved by rearranging
- The property $i = \sqrt{-1}$ is used
  - $\sqrt{-a} = \sqrt{a \times -1} = \sqrt{a} \times \sqrt{-1}$
- If the coefficients of the quadratic are real then the complex roots will occur in complex conjugate pairs
  - If $\boldsymbol{z = p + qi}$ **(**$\boldsymbol{q \neq 0}$**)** is a root of a quadratic with real coefficients then $\boldsymbol{z^* = p - qi}$ is also a root
- The **real part** of the solutions will have the same value as the x coordinate of the turning point on the graph of the quadratic
- When the coefficients of the quadratic equation are **non-real**, the solutions will **not** be complex conjugates
  - To solve these you can use the quadratic formula

How do we factorise a quadratic equation if it has complex roots?

- If we are given a quadratic equation in the form $az^2 + bz + c = 0$, where a, b, and $c \in \mathbb{R}$, $a \neq 0$ we can use its complex roots to write it in **factorised form**
  - Use the quadratic formula to find the two roots, $z = p + qi$ and $z^* = p - qi$
  - This means that $z - (p + qi)$ and $z - (p - qi)$ must both be factors of the quadratic equation
  - Therefore we can write $az^2 + bz + c = a(z - (p + qi))(z - (p - qi))$
  - This can be rearranged into the form $a(z - p - qi)(z - p + qi)$

How do we find a quadratic equation when given a complex root?

- If we are given a complex root in the form $z = p + qi$ we can find the quadratic equation in the form $az^2 + bz + c = 0$, where a, b, and $c \in \mathbb{R}$, $a \neq 0$
  - We know that the second root must be $z^* = p - qi$
  - This means that $z - (p + qi)$ and $z - (p - qi)$ must both be factors of the quadratic equation
  - Therefore we can write $az^2 + bz + c = (z - (p + qi))(z - (p - qi))$
  - Rewriting this as $((z - p) - qi))((z - p) + qi))$ makes expanding easier

- Expanding this gives the quadratic equation $z^2 - 2pz + (p^2 + q^2)$
  - $a = 1$
  - $b = -2p$
  - $c = p^2 + q^2$
- This demonstrates the important property $(x - z)(x - z^*) = x^2 - 2\text{Re}(z)x + |z|^2$

Worked example

a) Solve the quadratic equation $z^2 - 2z + 5 = 0$ and hence, factorise $z^2 - 2z + 5$.

Use the quadratic formula or completing the square to find the solutions.

Solutions of a quadratic equation	$ax^2+bx+c=0 \Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}, \ a\neq 0$

$a = 1$
$b = -2$
$c = 5$

$$z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm \sqrt{16}\sqrt{-1}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$z_1 = 1 + 2i \qquad z_2 = 1 - 2i$$

If the solutions are $z_1 = 1 + 2i$ and $z_2 = 1 - 2i$ then the factors must be $z - (1+2i)$ and $z - (1-2i)$

$$z^2 - 2z + 5 = (z - (1+2i))(z - (1-2i))$$

$$(z - 1 - 2i)(z - 1 + 2i)$$

b) Given that one root of a quadratic equation is $z = 2 - 3i$, find the quadratic equation in the form $az^2 + bz + c = 0$, where $a, b,$ and $c \in \mathbb{R}$, $a \neq 0$.

If $2-3i$ is one root then $2+3i$ must be the other root and the two factors must be $z-(2-3i)$ and $z-(2+3i)$.

$$(z-(2-3i))(z-(2+3i))=0$$

$$((z-2)+3i)((z-2)-3i)=0$$

$$(z-2)^2-(3i)^2=0$$

$$z^2-4z+4-9i^2=0$$

$i^2=-1 \therefore -9i^2=9$

$$z^2-4z+13=0$$

Complex Roots of Polynomials

How many roots should a polynomial have?

- We know that every **quadratic** equation has **two roots** (not necessarily distinct or real)
- This is a particular case of a more general rule:
  - Every polynomial equation of degree *n* has *n* roots
  - The *n* roots are not necessarily all **distinct** and therefore we need to count any **repeated** roots that may occur individually
- If a polynomial has **real coefficients**, then any **non-real roots** will occur as **complex conjugate pairs**
  - So if the polynomial has a non-real complex root, then it will always have the complex conjugate of that root as another root
- From the above rules we can state the following:
- A **cubic** equation of the form $ax^3 + bx^2 + cx + d = 0$ can have either:
  - 3 **real** roots
  - Or 1 **real** root and a complex **conjugate** pair
- A **quartic** equation of the form $ax^4 + bx^3 + cx^2 + dx + e = 0$ will have one of the following cases for roots:
  - 4 **real** roots
  - 2 **real** and 2 **nonreal** (a complex conjugate pair)
  - 4 **nonreal** (two complex conjugate pairs)

How do we solve a cubic equation with complex roots?

- Steps to solve a cubic equation with complex roots
  - If we are told that $p + qi$ is a root, then we know $p - qi$ is also a root
  - This means that $z - (p + qi)$ and $z - (p - qi)$ must both be factors of the cubic equation
  - Multiplying the above factors together gives us a quadratic factor of the form $(Az^2 + Bz + C)$
  - We need to find the third factor $(z - \alpha)$
  - **Multiply** the factors and **equate** to our original equation to get
    - $(Az^2 + Bz + C)(z - \alpha) = ax^3 + bx^2 + cx + d$
  - From there either
    - **Expand** and **compare coefficients** to find
    - Or use **polynomial division** to find the factor $(z - \alpha)$
  - Finally, write your **three roots** clearly

How do we solve a polynomial of any degree with complex roots?

- When asked to find the roots of any polynomial when we are given one, we use almost the same method as for a cubic equation
  - State the initial root and its conjugate and write their factors as a quadratic factor (as above) we will have two unknown roots to find, write these as factors $(z - \alpha)$ and $(z - \beta)$
  - The unknown factors also form a quadratic factor $(z - \alpha)(z - \beta)$
  - Then continue with the steps from above, either **comparing coefficients** or using **polynomial division**
    - If using polynomial division, then solve the quadratic factor you get to find the roots α and β

How do we solve polynomial equations with unknown coefficients?

- Steps to find unknown variables in a given equation when given a root:
  - **Substitute** the given root $p + qi$ into the equation $f(z) = 0$
  - **Expand** and **group** together the **real** and **imaginary** parts (these expressions will contain our unknown values)
  - **Solve** as simultaneous equations to find the unknowns
  - **Substitute** the values into the **original** equation
  - From here continue using the previously described methods for finding other roots for the polynomial

How do we factorise a polynomial when given a complex root?

- If we are given a root of a polynomial of any degree in the form $z = p + qi$
  - We know that the complex conjugate, $z^* = p - qi$ is another root
  - We can write $(z - (p + qi))$ and $(z - (p - qi))$ as two linear factors
    - Or rearrange into one quadratic factor
  - This can be multiplied out with another factor to find further factors of the polynomial
- For higher order polynomials more than one root may be given
  - If the further given root is complex then its complex conjugate will also be a root
  - This will allow you to find further factors

Worked example

Given that one root of a polynomial p(*x*) = $z\ \ + z\ \ - 7z + 65$ is 2 – 3i, find the other roots.

If $2-3i$ is one root then $2+3i$ must be the other root and two of the factors must be $z-(2-3i)$ and $z-(2+3i)$

Therefore a quadratic factor is $z^2 - 4z + 13$

There must exist a linear factor $(az+b)$

$\therefore (az+b)(z^2-4z+13) = z^3 + z^2 - 7z + 65$

Compare coefficients: $az^3 = z^3$ coefficient of z^3

$a = 1$

$13b = 65$ constant coefficient

$b = 5$

Therefore the factors are $z-(2-3i)$, $z-(2+3i)$ and $(z+5)$

$(z-(2-3i))(z-(2+3i))(z+5) = 0$

$z = (2 \pm 3i)$ and $z = -5$

1.9.4 De Moivre's Theorem

De Moivre's Theorem

What is De Moivre's Theorem?

- De Moivre's theorem can be used to find powers of complex numbers
- It states that for $z = r\text{ cis }\theta$, $z^n = [r(\cos\theta + \mathrm{i}\sin\theta)]^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$
  - Where
    - $z \neq 0$
    - r is the modulus, $|z|$, $r \in \mathbb{R}^+$
    - θ is the argument, arg z, $\theta \in \mathbb{R}$
    - $n \in \mathbb{R}$
- In Euler's form this is simply:
  - $(re^{\mathrm{i}\theta})^n = r^n e^{\mathrm{i}n\theta}$
- In words de Moivre's theorem tells us to raise the modulus by the power of n and multiply the argument by n
- In the formula booklet de Moivre's theorem is given in both polar and Euler's form:
  - $[r(\cos\theta + \mathrm{i}\sin\theta)]^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta) = r^n e^{\mathrm{i}n\theta} = r^n \text{ cis } n\theta$

How do I use de Moivre's Theorem to raise a complex number to a power?

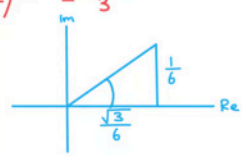
- If a complex number is in Cartesian form you will need to convert it to either modulus-argument (polar) form or exponential (Euler's) form first
  - This allows de Moivre's theorem to be used on the complex number
- You may need to convert it back to Cartesian form afterwards
- If a complex number is in the form $z = r(\cos(\theta) - \mathrm{i}\sin(\theta))$ then you will need to rewrite it as $z = r(\cos(-\theta) + \mathrm{i}\sin(-\theta))$ before applying de Moivre's theorem
- A useful case of de Moivre's theorem allows us to easily find the reciprocal of a complex number:
  - $\frac{1}{z} = \frac{1}{r}(\cos(-\theta) + \mathrm{i}\sin(-\theta)) = \frac{1}{r}e^{-\mathrm{i}\theta}$
  - Using the trig identities cos(-θ) = cos(θ) and sin(-θ) = - sin(θ) gives
  - $\frac{1}{z} = z^{-1} = r^{-1}[\cos(\theta) - \mathrm{i}\sin(\theta)] = \frac{1}{r}[\cos(\theta) - \mathrm{i}\sin(\theta)]$
- In general
  - $z^{-n} = r^{-n}[\cos(-n\theta) + \mathrm{i}\sin(-n\theta)] = r^{-n}[\cos(n\theta) - \mathrm{i}\sin(n\theta)]$

Worked example

Find the value of $\left(\frac{\sqrt{3}}{6}+\frac{1}{6}\mathrm{i}\right)^{-3}$, giving your answer in the form $a+bi$.

Write in Polar form: $r=\sqrt{\left(\frac{\sqrt{3}}{6}\right)^2+\left(\frac{1}{6}\right)^2}=\frac{1}{3}$

$\theta=\tan^{-1}\left(\frac{\frac{1}{6}}{\frac{\sqrt{3}}{6}}\right)=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$



$\left(\frac{\sqrt{3}}{6}+\frac{1}{6}i\right)^{-3}=\left(\frac{1}{3}\operatorname{cis}\left(\frac{\pi}{6}\right)\right)^{-3}$

Apply DeMoivre's Theorem: $\left(\frac{1}{3}\operatorname{cis}\left(\frac{\pi}{6}\right)\right)^{-3}=\left(\frac{1}{3}\right)^{-3}\operatorname{cis}\left(\frac{-3\pi}{6}\right)$

Convert back to Cartesian form:

$27\operatorname{cis}\left(-\frac{\pi}{2}\right)=27\left(\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right)$

$=27(0-i)$

$-27i$

Proof of De Moivre's Theorem

How is de Moivre's Theorem proved?

- When written in Euler's form the proof of de Moivre's theorem is easy to see:
  - Using the index law of brackets: $(re^{i\theta})^n = r^n e^{in\theta}$
- However Euler's form cannot be used to prove de Moivre's Theorem when it is in modulus-argument (polar) form
- **Proof by induction** can be used to prove de Moivre's Theorem for positive integers:
  - To prove de Moivre's Theorem for all positive integers, *n*
  - $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$
- STEP 1: Prove it is true for $n = 1$
  - $[r(\cos\theta + i\sin\theta)]^1 = r^1(\cos 1\theta + i\sin 1\theta) = r(\cos\theta + i\sin\theta)$
  - So de Moivre's Theorem is true for $n = 1$
- STEP 2: Assume it is true for $n = k$
  - $[r(\cos\theta + i\sin\theta)]^k = r^k(\cos k\theta + i\sin k\theta)$
- STEP 3: Show it is true for $n = k + 1$
  - $[r(\cos\theta + i\sin\theta)]^{k+1} = ([r(\cos\theta + i\sin\theta)]^k)([r(\cos\theta + i\sin\theta)]^1)$
  - According to the assumption this is equal to
    - $(r^k(\cos k\theta + i\sin k\theta))\,(r(\cos\theta + i\sin\theta))$
  - Using laws of indices and multiplying out the brackets:
    - $= r^{k+1}[\cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + i^2\sin k\theta\sin\theta]$
  - Letting $i^2 = -1$ and collecting the real and imaginary parts gives:
    - $= r^{k+1}[\cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)]$
  - Recognising that the real part is equivalent to $\cos(k\theta + \theta)$ and the imaginary part is equivalent to $\sin(k\theta + \theta)$ gives
    - $(r\,\text{cis}\,\theta)^{k+1} = r^{k+1}[\cos(k+1)\theta + i\sin(k+1)\theta]$
  - So de Moivre's Theorem is true for $n = k + 1$
- STEP 4: Write a conclusion to complete the proof
  - The statement is true for $n = 1$, and if it is true for $n = k$ it is also true for $n = k + 1$
  - Therefore, by the principle of mathematical induction, the result is true for all positive integers, *n*
- De Moivre's Theorem works for all real values of *n*
  - However you could only be asked to prove it is true for positive integers

Worked example

Show, using proof by mathematical induction, that for a complex number $z = r\,\text{cis}\theta$ and for all positive integers, n,

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

Step 1: Prove it is true for $n=1$

$$z^1 = [r(\cos\theta + i\sin\theta)]^1 = r^1(\cos 1\theta + i\sin 1\theta) = r(\cos\theta + i\sin\theta)$$

Step 2: Assume it is true for $n=k$

$$z^k = [r(\cos\theta + i\sin\theta)]^k = r^k(\cos k\theta + i\sin k\theta)$$

Step 3: Show it is true for $n=k+1$

$$z^{k+1} = [r(\cos\theta + i\sin\theta)]^{k+1}$$

Addition law of indices: $a^k a^1 = a^{k+1}$

$$= ([r(\cos\theta + i\sin\theta)]^k)([r(\cos\theta + i\sin\theta)]^1)$$

Using the assumption

$r^k \times r^1 = r^{k+1}$

$$= [r^k(\cos k\theta + i\sin k\theta)][r(\cos\theta + i\sin\theta)]$$

$$= r^{k+1}(\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$

$i^2 = -1$

$$= r^{k+1}[\cos k\theta\cos\theta + \cos k\theta(i\sin\theta) + \cos\theta(i\sin k\theta) + i^2\sin k\theta\sin\theta]$$

$$= r^{k+1}[\cos k\theta\cos\theta + i(\cos k\theta\sin\theta + \cos\theta\sin k\theta) - \sin k\theta\sin\theta]$$

Collect Re and Im parts

$$= r^{k+1}[\cos k\theta\cos\theta - \sin k\theta\sin\theta + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)]$$

$= \cos(k\theta + \theta)$ $= \sin(k\theta + \theta)$

$$= r^{k+1}[\cos(k\theta + \theta) + i\sin(k\theta + \theta)]$$

$$= r^{k+1}[\cos(k+1)\theta + i\sin(k+1)\theta]$$ so true for $n = k+1$

Step 4: Write a conclusion:

De Moivre's theorem is true for $n=1$, and if it is true for $n=k$ it is also true for $n=k+1$.
Therefore it is true for all $n \in \mathbb{Z}^+$

1.9.5 Roots of Complex Numbers

Roots of Complex Numbers

How do I find the square root of a complex number?

- The square roots of a complex number will themselves be complex:
  - i.e. if $z^2 = a + bi$ then $z = c + di$
- We can then square ($c + di$) and equate it to the original complex number ($a + bi$), as they both describe z^2:
  - $a + bi = (c + di)^2$
- Then expand and simplify:
  - $a + bi = c^2 + 2cdi + d^2i^2$
  - $a + bi = c^2 + 2cdi - d^2$
- As both sides are equal we are able to equate real and imaginary parts:
  - Equating the real components: $a = c^2 - d^2$ (1)
  - Equating the imaginary components: $b = 2cd$ (2)
- These equations can then be solved simultaneously to find the real and imaginary components of the square root
  - In general, we can rearrange (2) to make $\frac{b}{2d} = c$ and then substitute into (1)
  - This will lead to a quartic equation in terms of d; which can be solved by making a substitution to turn it into a quadratic
- The values of d can then be used to find the corresponding values of c, so we now have both components of both square roots ($c + di$)
- Note that one root will be the negative of the other root
  - g. $c + di$ and $-c - di$

How do I use de Moivre's Theorem to find roots of a complex number?

- De Moivre's Theorem states that a complex number in modulus-argument form can be raised to the power of *n* by
  - Raising the modulus to the power of *n* and multiplying the argument by *n*
- When in modulus-argument (polar) form de Moivre's Theorem can then be used to find the roots of a complex number by
  - Taking the *n*th root of the modulus and dividing the argument by *n*
  - If $z = r(\cos\theta + i\sin\theta)$ then $\sqrt[n]{z} = [r(\cos(\theta + 2\pi k) + i\sin(\theta + 2\pi k))]^{\frac{1}{n}}$
    - $k = 0, 1, 2, \ldots, n-1$
    - Recall that adding 2π to the argument of a complex number does not change the complex number

- Therefore we must consider how different arguments will give the same result
- This can be rewritten as $\sqrt[n]{z} = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + \mathrm{i}\sin\left(\frac{\theta + 2\pi k}{n}\right)\right)$

- This can be written in exponential (Euler's) form as
  - For $z^n = r\mathrm{e}^{\mathrm{i}\theta}$, $z = \sqrt[n]{r}\mathrm{e}^{\frac{\theta+2\pi k}{n}\mathrm{i}}$
- The *n*th root of complex number will have *n* roots with the properties:
  - The modulus is $\sqrt[n]{r}$ for all roots
  - There will be *n* different arguments spaced at equal intervals on the unit circle
  - This creates some geometrically beautiful results:
    - The five roots of a complex number raised to the power 5 will create a regular pentagon on an Argand diagram
    - The eight roots of a complex number raised to the power 8 will create a regular octagon on an Argand diagram
    - The *n* roots of a complex number raised to the power *n* will create a regular *n*-sided polygon on an Argand diagram
- Sometimes you may need to use your GDC to find the roots of a complex number
  - Using your GDC's store function will help when entering complicated modulus and arguments
  - Make sure you choose the correct form to enter your complex number in
  - Your GDC should be able to give you the answer in your preferred form

Worked example

a) Find the square roots of $5 + 12i$, giving your answers in the form $a + bi$.

Let $z^2 = 5 + 12i$, then $z = a + bi$

$z^2 = a^2 + 2abi + b^2 i^2$ ← $i^2 = -1$

$= a^2 + 2abi - b^2$

Therefore $5 + 12i = (a^2 - b^2) + 2abi$

Equate the real components: $a^2 - b^2 = 5$ ①

Equate the imaginary components: $2ab = 12$ ②

Solve the simultaneous equations: $a = \frac{6}{b} \Rightarrow \left(\frac{6}{b}\right)^2 - b^2 = 5$

$b^4 + 5b^2 - 36 = 0$

$(b^2 + 9)(b^2 - 4) = 0$

$b^2 = -9$ or $b^2 = 4$

no real solutions (for $b^2 = -9$)

$b = \pm 2$

$a = \pm 3$

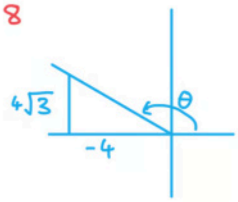
$z_1 = 3 + 2i$, $z_2 = -3 - 2i$

b) Solve the equation $z^3 = -4 + 4\sqrt{3}\,\mathrm{i}$ giving your answers in the form $r \operatorname{cis} \theta$.

Convert $-4+4\sqrt{3}i$ to Polar form:

$r = \sqrt{(-4)^2 + (4\sqrt{3})^2} = \sqrt{64} = 8$

$\theta = \pi - \left(\tan^{-1}\left(\frac{4\sqrt{3}}{4}\right)\right) = \frac{2\pi}{3}$



$\therefore \quad -4+4\sqrt{3}i = 8\,\text{cis}\left(\frac{2\pi}{3}\right)$

$z^3 = -4+4\sqrt{3}i$

$z = \sqrt[3]{-4+4\sqrt{3}i} = \left(8\,\text{cis}\left(\frac{2\pi}{3}\right)\right)^{\frac{1}{3}}$

$= \left(8^{\frac{1}{3}}\right)\text{cis}\left(\frac{\frac{2\pi}{3} + 2\pi k}{3}\right)$

Order 3 so there are 3 roots, use $k = 0, 1, 2$:

$z = 2\,\text{cis}\left(\frac{2\pi}{9}\right), \quad 2\,\text{cis}\left(\frac{8\pi}{9}\right), \quad 2\,\text{cis}\left(\frac{14\pi}{9}\right)$