DP IB Maths: AI HL

1.8 Eigenvalues & Eigenvectors

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1.8.1 Eigenvalues & Eigenvectors

Characteristic Polynomials

Eigenvalues and **eigenvectors** are properties of square matrices and are used in a lot of real-life applications including geometrical transformations and probability scenarios. In order to find these eigenvalues and eigenvectors, the **characteristic polynomial** for a matrix must be found and solved.

What is a characteristic polynomial?

- For a matrix $\boldsymbol{A}$, if $\boldsymbol{Ax} = \lambda\boldsymbol{x}$ when $\boldsymbol{x}$ is a non-zero vector and λ a **constant**, then λ is an **eigenvalue** of the matrix $\boldsymbol{A}$ and $\boldsymbol{x}$ is its corresponding **eigenvector**
- If $\boldsymbol{Ax} = \lambda\boldsymbol{x} \Rightarrow (\lambda\boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = 0$ or $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = 0$ and for $\boldsymbol{x}$ to be a non-zero vector, $\det(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0$
- The characteristic polynomial of an $n \times n$ matrix is:
$$p(\lambda) = \det(\lambda\boldsymbol{I} - \boldsymbol{A})$$
- In this course you will only be expected to find the characteristic equation for a 2×2 matrix and this will always be a **quadratic**

How do I find the characteristic polynomial?

- STEP 1
Write $\lambda\boldsymbol{I} - \boldsymbol{A}$, remembering that the identity matrix must be of the same order as $\boldsymbol{A}$
- STEP 2
Find the determinant of $\lambda\boldsymbol{I} - \boldsymbol{A}$ using the formula given to you in the formula booklet
$$\det \boldsymbol{A} = |\boldsymbol{A}| = ad - bc$$
- STEP 3
Re-write as a polynomial

Worked example

Find the characteristic polynomial of the following matrix

$$A = \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}.$$

$p(\lambda) = \det(\lambda I - A)$

$= \det\left(\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}\right)$

$= \det\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}\right)$

$= \det\begin{pmatrix} \lambda-5 & -4 \\ -3 & \lambda-1 \end{pmatrix}$

Determinant of a 2×2 matrix	$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = \lvert A \rvert = ad - bc$

$= (\lambda-5)(\lambda-1) - (-4)(-3)$

$= \lambda^2 - 5\lambda - \lambda + 5 - 12$

$\boxed{p(\lambda) = \lambda^2 - 6\lambda - 7}$

Eigenvalues & Eigenvectors

How do you find the eigenvalues of a matrix?

- The eigenvalues of matrix $\boldsymbol{A}$ are found by solving the **characteristic polynomial** of the matrix
- For this course, as the characteristic polynomial will always be a **quadratic**, the polynomial will always generate one of the following:
  - **two real and distinct** eigenvalues,
  - **one real repeated** eigenvalue or
  - **complex** eigenvalues

How do you find the eigenvectors of a matrix?

- A value for $\boldsymbol{x}$ that satisfies the equation is an **eigenvector** of matrix $\boldsymbol{A}$
- Any scalar multiple of $\boldsymbol{x}$ will also satisfy the equation and therefore there an **infinite number** of eigenvectors that correspond to a particular eigenvalue
- STEP 1

  Write $\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix}$
- STEP 2

  Substitute the eigenvalues into the equation $(\lambda \boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \boldsymbol{0}$, and form two equations in terms of x and y
- STEP 3

  There will be an infinite number of solutions to the equations, so choose one by letting one of the variables be equal to 1 and using that to find the other variable

Worked example

Find the eigenvalues and associated eigenvectors for the following matrices.

a) $$A = \begin{pmatrix} 5 & 4 \\ 3 & 1 \end{pmatrix}.$$

Solve the characteristic polynomial to find the eigenvalues

$p(\lambda) = \lambda^2 - 6\lambda - 7$ ← From worked example above in Characteristic Polynomials

$(\lambda - 7)(\lambda + 1)$

$\Rightarrow \boxed{\lambda = 7}$ $\boxed{\lambda = -1}$

Use the eigenvalues in the equation $(\lambda I - A)\ x = 0$ to find the eigenvectors

For $\lambda = 7 \Rightarrow \left(7\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \begin{pmatrix}5 & 4\\3 & 1\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$$\left(\begin{pmatrix}7 & 0\\0 & 7\end{pmatrix} - \begin{pmatrix}5 & 4\\3 & 1\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}2 & -4\\-3 & 6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\left.\begin{aligned}2x - 4y &= 0\\-3x + 6y &= 0\end{aligned}\right\}\ 2y = x$$

The eigenvector associated with $\lambda = 7$ is any multiple of $\begin{pmatrix}2\\1\end{pmatrix}$

For $\lambda = -1 \Rightarrow \left(-1\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \begin{pmatrix}5 & 4\\3 & 1\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$$\left(\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix} - \begin{pmatrix}5 & 4\\3 & 1\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}-6 & -4\\-3 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\left.\begin{aligned}-6x - 4y &= 0\\-3x - 2y &= 0\end{aligned}\right\}\ 2y = -3x$$

The eigenvector associated with $\lambda = -1$ is any multiple of $\begin{pmatrix}2\\-3\end{pmatrix}$

b) $\boldsymbol{B} = \begin{pmatrix}1 & -5\\2 & 3\end{pmatrix}$.

Find the characteristic polynomial

$$p(\lambda) = \det\begin{pmatrix} \lambda-1 & 5 \\ -2 & \lambda-3 \end{pmatrix}$$

$$= (\lambda-1)(\lambda-3) - (5)(-2)$$

$$= \lambda^2 - 3\lambda - \lambda + 3 + 10$$

$$\boxed{p(\lambda) = \lambda^2 - 4\lambda + 13}$$

Solve the characteristic polynomial to find the eigenvalues by hand or using the GDC

$$p(\lambda) = \lambda^2 - 4\lambda + 13 = 0$$

$$(\lambda-2)^2 - 4 + 13 = 0$$

$$(\lambda-2)^2 = -9$$

$$\lambda = 2 \pm \sqrt{-9}$$

$$\boxed{\lambda = 2 \pm 3i}$$

Use the eigenvalues in the equation $(\lambda I - A)\,x = 0$ to find the eigenvectors

For $\lambda = 2+3i \Rightarrow \left((2+3i)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \begin{pmatrix}1 & -5\\2 & 3\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$$\left(\begin{pmatrix}2+3i & 0\\0 & 2+3i\end{pmatrix} - \begin{pmatrix}1 & -5\\2 & 3\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}1+3i & 5\\-2 & -1+3i\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\left.\begin{aligned}(1+3i)x + 5y &= 0\\-2x + (-1+3i)y &= 0\end{aligned}\right\} 2x = (-1+3i)y$$

Both equations can be simplified to the same thing

The eigenvector associated with $\lambda = 2+3i$ is any multiple of $\begin{pmatrix}-1+3i\\2\end{pmatrix}$

For $\lambda = 2-3i \Rightarrow \left((2-3i)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \begin{pmatrix}1 & -5\\2 & 3\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$$\left(\begin{pmatrix}2-3i & 0\\0 & 2-3i\end{pmatrix} - \begin{pmatrix}1 & -5\\2 & 3\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}1-3i & 5\\-2 & -1-3i\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\left.\begin{aligned}(1-3i)x + 5y &= 0\\-2x + (-1-3i)y &= 0\end{aligned}\right\} 2x = (-1-3i)y$$

Both equations can be simplified to the same thing

The eigenvector associated with $\lambda = 2-3i$ is any multiple of $\begin{pmatrix}-1-3i\\2\end{pmatrix}$

1.8.2 Applications of Matrices

Diagonalisation

What is matrix diagonalisation?

- A **non-zero, square** matrix is considered to be **diagonal** if all elements **not** along its leading diagonal are **zero**
- A matrix $\boldsymbol{P}$ can be said to diagonalise matrix $\boldsymbol{M}$, if $\boldsymbol{D}$ is a diagonal matrix where $\boldsymbol{D} = \boldsymbol{P}^{-1}\boldsymbol{M}\boldsymbol{P}$
- If matrix $\boldsymbol{M}$ has **eigenvalues** λ_1, λ_2 and **eigenvectors** $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and is diagonisable by $\boldsymbol{P}$, then
  - $\boldsymbol{P} = (\boldsymbol{x}_1\, \boldsymbol{x}_2)$, where the first column is the eigenvector $\boldsymbol{x}_1$ and the second column is the eigenvector $\boldsymbol{x}_2$
  - $\boldsymbol{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$
- You will only need to be able to diagonalise 2×2 matrices
- You will only need to consider matrices with **real**, **distinct eigenvalues**
  - If there is only one eigenvalue, the matrix is either already diagonalised or cannot be diagonalised
  - Diagonalisation of matrices with complex or imaginary eigenvalues is outside the scope of the course

Worked example

The matrix $\boldsymbol{M}=\begin{pmatrix}5 & 4\\ 3 & 1\end{pmatrix}$ has the eigenvalues $\lambda_1=7$ and $\lambda_2=-1$ with eigenvectors $\boldsymbol{x}_1=\begin{pmatrix}2\\ 1\end{pmatrix}$ and $\boldsymbol{x}_2=\begin{pmatrix}2\\ -3\end{pmatrix}$ respectively.

Show that $\boldsymbol{P}_1=(\boldsymbol{x}_1\,\boldsymbol{x}_2)$ and $\boldsymbol{P}_2=(\boldsymbol{x}_2\,\boldsymbol{x}_1)$ both diagonalise $\boldsymbol{M}$.

Show that $P^{-1}MP$ produces a diagonal matrix

Inverse of a 2×2 matrix	$A=\begin{pmatrix}a & b\\ c & d\end{pmatrix} \Rightarrow A^{-1}=\frac{1}{\det A}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix},\ ad\neq bc$

$$P_1=\begin{pmatrix}2 & 2\\ 1 & -3\end{pmatrix} \Rightarrow P_1^{-1}=-\frac{1}{8}\begin{pmatrix}-3 & -2\\ -1 & 2\end{pmatrix}=\frac{1}{8}\begin{pmatrix}3 & 2\\ 1 & -2\end{pmatrix}$$

$$D_1=P_1^{-1}MP_1=\frac{1}{8}\begin{pmatrix}3 & 2\\ 1 & -2\end{pmatrix}\begin{pmatrix}5 & 4\\ 3 & 1\end{pmatrix}\begin{pmatrix}2 & 2\\ 1 & -3\end{pmatrix}$$

$$=\frac{1}{8}\begin{pmatrix}3 & 2\\ 1 & -2\end{pmatrix}\begin{pmatrix}14 & -2\\ 7 & 3\end{pmatrix}$$

$$=\frac{1}{8}\begin{pmatrix}56 & 0\\ 0 & -8\end{pmatrix}$$

$$\boxed{D=\begin{pmatrix}7 & 0\\ 0 & -1\end{pmatrix}}$$

D = Diagonal matrix of eigenvalues

$$P_2=\begin{pmatrix}2 & 2\\ -3 & 1\end{pmatrix} \Rightarrow P_2^{-1}=\frac{1}{8}\begin{pmatrix}1 & -2\\ 3 & 2\end{pmatrix}$$

$$D_2=P_2^{-1}MP_2=\frac{1}{8}\begin{pmatrix}1 & -2\\ 3 & 2\end{pmatrix}\begin{pmatrix}5 & 4\\ 3 & 1\end{pmatrix}\begin{pmatrix}2 & 2\\ -3 & 1\end{pmatrix}$$

$$=\frac{1}{8}\begin{pmatrix}1 & -2\\ 3 & 2\end{pmatrix}\begin{pmatrix}-2 & 14\\ 3 & 7\end{pmatrix}$$

$$=\frac{1}{8}\begin{pmatrix}-8 & 0\\ 0 & 56\end{pmatrix}$$

$$D=\boxed{\begin{pmatrix}-1 & 0\\ 0 & 7\end{pmatrix}}$$

D = Diagonal matrix of eigenvalues

Matrix Powers

One of the main applications of diagonalising a matrix is to make it easy to find **powers** of the matrix, which is useful when modelling transient situations such as the movement of populations between two towns.

How can the diagonalised matrix be used to find higher powers of the original matrix?

- The equation to find the diagonalised matrix can be re-arranged for $\boldsymbol{M}$:

$$\boldsymbol{D} = \boldsymbol{P}^{-1}\boldsymbol{M}\boldsymbol{P} \Rightarrow \boldsymbol{M} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$$

- Finding higher powers of a matrix when it is diagonalised is straight forward:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$$

- Therefore, we can easily find higher powers of the matrix using the **power formula** for a matrix found in the formula booklet:

$$\boldsymbol{M}^n = \boldsymbol{P}\boldsymbol{D}^n\boldsymbol{P}^{-1}$$

Worked example

The matrix $\boldsymbol{M}=\begin{pmatrix}3 & -2\\ -4 & 1\end{pmatrix}$ has the eigenvalues $\lambda_1=-1$ and $\lambda_2=5$ with eigenvectors $\boldsymbol{x}_1=\begin{pmatrix}1\\2\end{pmatrix}$ and $\boldsymbol{x}_2=\begin{pmatrix}1\\-1\end{pmatrix}$ respectively.

a) Show that $\boldsymbol{M}^n$ can be expressed as

$$\boldsymbol{M}^n=-\frac{1}{3}\begin{pmatrix}(-(-1)^n-2(5)^n) & (-(-1)^n+(5)^n)\\ (-2(-1)^n+2(5)^n) & (-2(-1)^n-(5)^n)\end{pmatrix}$$

Find D, P and P^{-1}

$$D=\begin{pmatrix}-1 & 0\\ 0 & 5\end{pmatrix} \quad P=\begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix} \Rightarrow P^{-1}=-\frac{1}{3}\begin{pmatrix}-1 & -1\\ -2 & 1\end{pmatrix}$$

Use the matrix power formula from the formula booklet

Power formula for a matrix	$M^n=PD^nP^{-1}$	P is the matrix of eigenvectors and D is the diagonal matrix of eigenvalues

$$M^n=-\frac{1}{3}\begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 5\end{pmatrix}^n\begin{pmatrix}-1 & -1\\ -2 & 1\end{pmatrix}$$

$$=-\frac{1}{3}\begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}\begin{pmatrix}-1^n & 0\\ 0 & 5^n\end{pmatrix}\begin{pmatrix}-1 & -1\\ -2 & 1\end{pmatrix}$$

When multiplying these expressions, be careful! $-2\times 5^n=-2(5^n)$ NOT -10^n

$$=-\frac{1}{3}\begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}\begin{pmatrix}-(-1)^n & -(-1)^n\\ -2(5)^n & (5)^n\end{pmatrix}$$

$$\boxed{M^n=-\frac{1}{3}\begin{pmatrix}(-(-1)^n-2(5)^n) & (-(-1)^n+(5)^n)\\ (-2(-1)^n+2(5)^n) & (-2(-1)^n-(5)^n)\end{pmatrix}}$$

b) Hence find $\boldsymbol{M}^5$.

Substitute $n = 5$

$$M^5 = -\frac{1}{3}\begin{pmatrix} (-(-1)^5 - 2(5)^5) & (-(-1)^5 + (5)^5) \\ (-2(-1)^5 + 2(5)^5) & (-2(-1)^5 - (5)^5) \end{pmatrix}$$

$$= -\frac{1}{3}\begin{pmatrix} -6249 & 3126 \\ 6252 & -3123 \end{pmatrix}$$

$$\boxed{M^5 = \begin{pmatrix} 2083 & -1042 \\ -2084 & 1041 \end{pmatrix}}$$