DP IB Maths: AA HL

1.7 Permutations & Combinations

Contents

1.7.1 Counting Principles

Counting Principles

What is meant by counting principles?

- The fundamental counting principle states that if there are m ways to do one thing and n ways to do another there are $m \times n$ ways to do **both** things
- Applying counting principles allows us to…
  - … analyse patterns and make generalisations about real work situations
  - … find the number of **permutations** of n items
  - … find the number of ways of choosing an item from a list of n items
  - … find the number of ways of choosing r items from n items
  - … find the number of ways of permutating r items from n items
- The topic of counting principles is a particularly interesting part of mathematics that can lead to the development of working with very large numbers
- It is always vital to consider whether objects taken from each list can be repeated or not
  - For example a four digit PIN from ten numbers where each number can be used repeatedly would be $10 \times 10 \times 10 \times 10$
    - There are 10 options for the first and ten options for the second number and so on
  - If the numbers could only be used once then the number of options for each digit would **reduce** with each digit
    - There are 10 options for the first, nine options for the second, eight for the third and so on
    - This concept will be explored further in the **permutations** revision note

How do I choose an item from a list of *m* items AND another item from a list of *n* items?

- If a question requires you to choose an item from one list **AND** an item from another list you should **multiply** the number of options in each list
  - In general if you see the word 'AND' you will most likely need to 'MULTIPLY'
- For example if you are choosing a pen and a pencil from 4 pens and 5 pencils:
  - You can choose 1 item from 4 pens AND 1 item from 5 pencils
  - You will have 4×5 different options to choose from

How do I choose an item from a list of *m* items OR another item from a list of *n* items?

- If a question requires you to choose an item from one list **OR** an item from another list you should **add** the number of options in each list
  - In general if you see the word 'OR' you will most likely need to 'ADD'
- For example if you are choosing a pen or a pencil from 4 pens and 5 pencils:
  - You can choose 1 item from 4 pens OR 1 item from 5 pencils
  - You will have $4 + 5$ different options to choose from

Worked example

Harry is going to a formal event and is choosing what accessories to add to his outfit. He has seven different ties, four different bow ties and five different pairs of cufflinks. How many different ways can Harry get ready if he chooses:

a) Either a tie, a bow tie or a pair of cufflinks?

Harry has $7 + 4 + 5$ different items to choose from

He wants a tie OR a bow tie OR a pair of cufflinks

OR means ADD $7 + 4 + 5 = 16$

$\boxed{\text{16 different ways}}$

b) A pair of cufflinks and either a tie or a bow tie?

He wants a tie AND a pair of cufflinks

OR a bow tie AND a pair of cufflinks

AND means MULTIPLY

A tie AND cufflinks $= 7 \times 5 = 35$ ways

A bowtie AND cufflinks $= 4 \times 5 = 20$ ways

OR means ADD $35 + 20 = 55$ ways

$\boxed{\text{55 different ways}}$

1.7.2 Permutations & Combinations

Permutations

What are Permutations?

- A **permutation** is the number of possible arrangements of a set of objects when the **order** of the arrangements matters
- A permutation can either be finding the number of ways to arrange *n* items or finding the number of ways to arrange *r* out of *n* items

How many ways can *n* different objects be arranged?

- When considering how many ways you can arrange a number of **different** objects in a row consider how many of the objects can go in the first position, how many can go in the second and so on
- For *n* = 2 there are two options for the first position and then there will only be one option to go in the second position so:
  - The first object has **two** places it could go and the second object has **one** place
  - By the fundamental counting principle **both** objects have 2 × 1 places to go
  - For example to arrange the letters A and B we have
    - AB and BA
- For *n* = 3 there are three options for the first position and then there will be two options for the second position and one for the third position so
  - The first object has **three** places it could go, the second object has **two** places and the third object has **one** place
  - By the fundamental counting principle **the three** objects have 3 × 2 × 1 places to go
  - For example to arrange the letters A, B and C we have
    - ABC, ACB, BAC, BCA, CAB and CBA
- For *n* objects there are *n* options for the first position, *n* - 1 options for the second position and so on until there is only one object left to go in final position
- The number of **permutations** of *n* different objects is *n* factorial (*n*!)
  - Where $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$
  - For 5 **different** items there are $5! = 5 \times 4 \times 3 \times 2 = 120$ permutations
  - For 6 **different** items there are $6! = 6 \times 5 \times 4 \times 3 \times 2 = 720$ permutations
  - It is easy to see how quickly the number of possible permutations of different items can increase
  - For 10 different items there are 10! = 3 628 800 possible permutations

What are factorials?

- Factorials are a type of mathematical operation (just like +, -, ×, ÷)
- The symbol for factorial is !
  - So to take a factorial of any non-negative integer, *n*, it will be written *n*! and pronounced '*n factorial*'
- The factorial function for any positive integer, *n*, is $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$
  - For example, 5 factorial is $5! = 5 \times 4 \times 3 \times 2 \times 1$

- The factorial of a negative number is not defined
  - You cannot arrange a negative number of items
- 0! = 1
  - There are no positive integers less than zero, so zero items can only be arranged once
- Your GDC will have a mode for calculating factorials, make sure you can put yours into the correct mode
- Most normal calculators cannot handle numbers greater than about 70!, experiment with yours to see the greatest value of *x* such that your calculator can handle *x*!

What are the key properties of using factorials?

- Some important relationships to be aware of are:
  - $n! = n \times (n-1)!$
    - Therefore $\frac{n!}{(n-1)!} = n$
  - $n! = n \times (n-1) \times (n-2)!$
    - Therefore $\frac{n!}{(n-2)!} = n \times (n-1)$
- Expressions with factorials in can be simplified by considering which values cancel out in the fraction
  - Dividing a large factorial by a smaller one allows many values to cancel out
    - $\frac{8!}{5!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = 8 \times 7 \times 6$

How do we find *r* permutations of *n* items?

- If we only want to find the number of ways to arrange a few out of *n* different objects, we should consider how many of the objects can go in the first position, how many can go in the second and so on
- If we wanted to arrange 3 out of 5 different objects, then we would have 3 positions to place the objects in, but we would have 5 options for the first position, 4 for the second and 3 for the third
  - This would be 5 × 4 × 3 ways of permutating 3 out of 5 different objects
  - This is equivalent to $\frac{5!}{2!} = \frac{5!}{(5-3)!}$
- If we wanted to arrange 4 out of 10 different objects, then we would have 4 positions to place the objects in, but we would have 10 options for the first position, 9 for the second, 8 for the third and 7 for the fourth
  - This would be 10 × 9 × 8 × 7 ways of permutating 4 out of 10 different objects
  - This is equivalent to $\frac{10!}{6!} = \frac{10!}{(10-4)!}$
- If we wanted to arrange *r* out of *n* different objects, then we would have *r* positions to place the objects in, but we would have *n* options for the first position, $(n-1)$ for the second, $(n-2)$ for the third and so on until we reach $(n-(r-1))$
  - This would be $n \times (n-1) \times \ldots \times (n-r+1)$ ways of permutating r out of n different objects

    - This is equivalent to $\frac{n!}{(n-r)!}$
- The function $\frac{n!}{(n-r)!}$ can be written as ${}^{n}\mathrm{P}_{r}$
    - Make sure you can find and use this button on your calculator
- The same function works if we have *n* spaces into which we want to arrange *r* objects, consider
    - for example arranging five people into a row of ten empty chairs

Permutations when two or more items must be together

- If two or more items must stay together within an arrangement, it is easiest to think of these items as 'stuck' together
- These items will become one within the arrangement
- Arrange this 'one' item with the others as normal
- Arrange the items within this 'one' item separately
- Multiply these two arrangements together

Permutations when two or more items cannot be all together

- If **two** items must be **separated …**
    - consider the number of ways these two items would be together
    - subtract this from the total number of arrangements without restrictions
- If **more than two** items must be separated…
    - consider whether all of them must be **completely separate** (none can be next to each other) or whether they **cannot all be together** (but two could still be next to each other)
    - If they **cannot all be together** then we can treat it the same way as separating two items and subtract the number of ways they would all be together from the total number of permutations of the items, the final answer will include all permutations where two items are still together
    - If the items must all be **completely separate** then
        - lay out the rest of the items in a line with a space in between each of them where one of the items which cannot be together could go
        - remember that this could also include the space before the first and after the last item
        - You would then be able to fit the items which cannot be together into any of these spaces, using the *r* permutations of *n* items rule $\left({}^{n}\mathrm{P}_{r}\right)$
        - You do not need to fill every space

Permutations when two or more items must be in specific places

- Most commonly this would be arranging a word where specific letters would go in the first and last place
- Or arranging objects where specific items have to be at the ends/in the middle
    - Imagine these specific items are stuck in place, then you can find the number of ways to arrange the rest of the items around these 'stuck' items
- Sometimes the items must be grouped
    - for example all vowels must be before the consonants

- Or all the red objects must be on one side and the blue objects must be on the other
- Find the number of permutations within each group separately and multiply them together
- Be careful to check whether the groups could be in either place
  - e.g. the vowels on one side and consonants on the other
  - or if they must be in specific places (the vowels **before** the consonants)
- If the groups could be in either place than your answer would be multiplied by two
- If there were n groups that could be in any order then your answer would be multiplied by $n!$

Worked example

Find the number of ways nine different tasks can be carried out given that two particular tasks must not be carried out consecutively.

Start by considering the tasks that can be carried out consecutively:

$T_1 \quad T_2 \quad T_3 \quad T_4 \quad T_5 \quad T_6 \quad T_7$

There are 7! ways of carrying out the other seven tasks.

Then consider the tasks with restrictions:

$\times\, T_1 \,\times\, T_2 \,\times\, T_3 \,\times\, T_4 \,\times\, T_5 \,\times\, T_6 \,\times\, T_7 \,\times$ positions where T_8 or T_9 could go

There are 8 positions in which the two tasks could go but only 2 tasks to fill the spaces $= 8 \times 7$ options

$= 8P2$

(8 options for T_8 and 7 options for T_9)

$$\boxed{\text{Total} = 7! \times 8 \times 7 = 282240 \text{ ways}}$$

Alternative method:

Put the two tasks together and then subtract from the total.

$T_1 \quad T_2 \quad T_3 \quad T_4 \quad T_5 \quad T_6 \quad T_7 \quad T_8 \; T_9$

The two tasks are 'stuck' together.

number of ways without restrictions

$$9! - (8! \times 2!)$$

7 'free' tasks plus the two tasks 'stuck' together.

There are 2! ways of carrying out the two 'stuck' tasks

Combinations

What is the difference between permutations and combinations?

- A **combination** is the number of possible arrangements of a set of objects when the **order** of the arrangements **does not matter**
  - On the other hand a **permutation** is when the order of arrangement **does matter**
- A combination will be finding the number of ways to **choose** *r* out of *n* items
  - The order in which the *r* items are chosen is not important
  - For example if we are choosing two letters from the word CAB, AB and BA would be considered the same combination but different permutations

How do we find *r* combinations of *n* items?

- If we want to find the number of ways to **choose** 2 out of 3 different objects, but we don't mind the order in which they are chosen, then we could find the number of **permutations of 2 items from 3** and then divide by the number of ways of arranging each combination
  - For example if we want to choose 2 letters from *A, B* and *C*
    - There are 6 permutations of 2 letters:
    - *AB, BA, AC, CA, BC, CB*
    - For each combination of 2 letters there are 2 (2 × 1) ways of arranging them
    - (for example, *AB* and *BA*)
    - So divide the total number of permutations (6) by the number of ways of arranging each combination (2) to get 3 combinations
- If we want to find the number of ways to **choose** 3 out of 5 different objects, but we don't mind the order in which they are chosen, then we could find the number of **permutations of 3 items from *5*** and then divide by the number of ways of arranging each combination
  - For example if we want to choose 3 letters from *A, B, C, D* and *E*
    - There are 60 permutations of 3 letters:
    - *ABC, ACB, BAC, BCA, CAB, CBA, ABD, ADB*, etc
    - For each combination of 3 letters there are 6 (3 × 2 ×1) ways of arranging them (for example, *ABC, ACB, BAC, BCA, CAB* and *CBA*)
    - So divide the total number of permutations (60) by the number of ways of arranging each combination (3! = 6) to get 10 combinations
- If we want to find the number of ways to **choose** *r* items out of *n* different objects, but we don't mind the order in which they are chosen, then we could find the number of **permutations of *r* items from *n*** and then divide by the number of ways of arranging each combination
- Recall that the formula for *r* permutations of *n* items is
  - $^{n}\mathrm{P}_{r} = \dfrac{n!}{(n-r)!}$
- This would include *r*! ways of repeating each combination
- The formula for *r* **combinations** of *n* items is
  - $\dfrac{^{n}\mathrm{P}_{r}}{r!} = \dfrac{n!}{(n-r)!\,r!}$

- The function $\dfrac{n!}{(n-r)!\,r!}$ can be written as ${}^{n}\mathrm{C}_{r}$ or $\begin{pmatrix} n \\ r \end{pmatrix}$ and is often read as '*n choose r*'
  - Make sure you can find and use this button on your calculator
- The formulae for permutations and combinations satisfy the following relationship:
  - ${}^{n}\mathrm{C}_{r} = \dfrac{{}^{n}\mathrm{P}_{r}}{r!}$
- The formula ${}^{n}\mathrm{C}_{r} = \dfrac{n!}{r!(n-r)!}$ can be found in the formula booklet

What do I need to know about combinations?

- The formula ${}^{n}C_{r} = \dfrac{n!}{(n-r)!r!}$ is also known as a **binomial** coefficient
- ${}^{n}C_{n} = {}^{n}C_{0} = 1$
  - It is easy to see that there is only one way of arranging *n* objects out of *n* and also there can only be one way of arranging 0 objects out of *n*
  - By considering the formula for this, it reinforces the fact that 0! Must equal 1
- The binomial coefficients are symmetrical, so ${}^{n}C_{r} = {}^{n}C_{n-r}$
  - This can be seen by considering the formula for ${}^{n}C_{r}$
  - ${}^{n}\mathrm{C}_{n-r} = \dfrac{n!}{(n-r)!(n-(n-r))!} = \dfrac{n!}{r!(n-r)!} = {}^{n}\mathrm{C}_{r}$

How do I know when to multiply or add?

- Many questions will ask you to find combinations of a group of different items from a bigger group of a specified number of those different items
  - For example, find the number of ways five questions could be chosen from a bank of twenty different pure and ten different statistics questions
  - The hint in this example is the word 'chosen', this tells you that the order in which the questions are chosen doesn't matter
- Sometimes questions will have restrictions,
  - For example there should be three pure and two statistics chosen from the bank of questions,
  - Or there must be at least two pure questions within the group
- If unsure about whether to add or multiply your options, ask yourself if A **and** B are both needed, or if A **or** B is needed
  - Always **multiply** if the answer is **and,** and **add** if the answer is **or**
  - For example if we needed exactly three pure **and** two statistics questions we would find the amount of each and multiply them

Worked example

Oscar has to choose four books from a reading list to take home over the summer. There are four fantasy books, five historical fiction books and two classics available for him to choose from. Find the number of ways that Oscar can choose four books if he decides to have:

i) Two fantasy books and two historical fictions.

Choosing two fantasy from four: $^4C_2$

Choosing two historical fiction from five: $^5C_2$

Total $= {}^4C_2 \times {}^5C_2 = 6 \times 10$

$\boxed{\text{60 options}}$

ii) At least one of each type of book.

To choose two of one and one of each of the others:
Let F represent a fantasy book, H represent historical fiction and C represent a classic:

these are the numbers of options Oscar can choose from

	F	H	C	
	4	5	2	
these are the choices he can make	1	1	2	$^4C_1 \times {}^5C_1 \times {}^2C_2 = 4 \times 5 \times 1$
	1	2	1	$^4C_1 \times {}^5C_2 \times {}^2C_1 = 4 \times 10 \times 2$
	2	1	1	$^4C_2 \times {}^5C_1 \times {}^2C_1 = 6 \times 5 \times 2$

'and' so multiply.

Oscar can choose from one of these options so we add them:

$20 + 80 + 60$

'or' so add

$\boxed{\text{160 options}}$

iii) At least two fantasy books.

Oscar can choose two, three or four fantasy books. It does not matter whether the other books are classics or historical fiction.

This is just 'Not Fantasy' (5H + 2C)

F	NF	
2	2	$4C2 \times 7C2 = 6 \times 21 = 126$
3	1	$4C3 \times 7C1 = 4 \times 7 = 28$
4	0	$4C4 \times 7C0 = 1 \times 1 = 1$

Use a system to make sure you don't forget any. Check that all the options add up to 4.

Total number of options = 126 + 28 + 1

155 Options