DP IB Maths: AI HL

1.7 Matrices

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1.7.1 Introduction to Matrices

Introduction to Matrices

Matrices are a useful way to represent and manipulate data in order to model situations. The elements in a matrix can represent data, equations or systems and have many real-life applications.

What are matrices?

- A matrix is a **rectangular array** of **elements** (numerical or algebraic) that are arranged in **rows** and **columns**
- The **order** of a matrix is defined by the **number** of rows and columns that it has
  - The order of a matrix with m rows and n columns is $m \times n$
- A matrix $\boldsymbol{A}$ can be defined by $\boldsymbol{A} = (a_{ij})$ where $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$ and a_{ij} refers to the element in row i, column j

$$\boldsymbol{A} = (a_{i,j}) = \left.\overbrace{\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix}}^{\text{Number of columns, } n = 3}\right\} \text{Number of rows, } m = 2$$

What type of matrices are there?

- A **column matrix** (or column vector) is a matrix with a **single column**, $n = 1$
- A **row matrix** is a matrix with a **single row**, $m = 1$
- A **square matrix** is one in which the number of rows is **equal** to the number of columns, $m = n$
- Two matrices are **equal** when they are of the **same order** and their **corresponding elements** are **equal**, i.e. $a_{ij} = b_{ij}$ for all elements
- A **zero matrix**, $\boldsymbol{O}$, is a matrix in which all the elements are 0, e.g. $\boldsymbol{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
- The identity matrix, $\boldsymbol{I}$, is a **square** matrix in which all elements along the **leading diagonal** are 1 and the rest are 0, e.g. $\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Worked example

Let the matrix $\boldsymbol{A} = \begin{pmatrix} 5 & -3 & 7 \\ -1 & 2 & 4 \end{pmatrix}$

a) Write down the order of $\boldsymbol{A}$.

A is a 2×3 Matrix

b) State the value of $a_{2,3}$.

$a_{23} = 4$

1.7.2 Operations with Matrices

Matrix Addition & Subtraction

Just as with ordinary numbers, **matrices** can be **added** together and **subtracted** from one another, provided that they meet certain conditions.

How is addition and subtraction performed with matrices?

- Two matrices of the **same order** can be added or subtracted
- Only **corresponding elements** of the two matrices are added or subtracted
  - $\boldsymbol{A} \pm \boldsymbol{B} = (a_{ij}) \pm (b_{ij}) = (a_{ij} \pm b_{ij})$
- The **resultant** matrix is of the **same order** as the original matrices being added or subtracted

What are the properties of matrix addition and subtraction?

- $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$ (commutative)
- $\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$ (associative)
- $\boldsymbol{A} + \boldsymbol{O} = \boldsymbol{A}$
- $\boldsymbol{O} - \boldsymbol{A} = -\boldsymbol{A}$
- $\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + (-\boldsymbol{B})$

Worked example

Consider the matrices $\boldsymbol{A} = \begin{pmatrix} -4 & 2 \\ 7 & 3 \\ 1 & -5 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 2 & 6 \\ 5 & -9 \\ -2 & -3 \end{pmatrix}$.

a) Find $\boldsymbol{A} + \boldsymbol{B}$.

$$A + B = \begin{pmatrix} -4 & 2 \\ 7 & 3 \\ 1 & -5 \end{pmatrix} + \begin{pmatrix} 2 & 6 \\ 5 & -9 \\ -2 & -3 \end{pmatrix} = \boxed{\begin{pmatrix} -2 & 8 \\ 12 & -6 \\ -1 & -8 \end{pmatrix}}$$

b) Find $\boldsymbol{A} - \boldsymbol{B}$.

$$A - B = \begin{pmatrix} -4 & 2 \\ 7 & 3 \\ 1 & -5 \end{pmatrix} - \begin{pmatrix} 2 & 6 \\ 5 & -9 \\ -2 & -3 \end{pmatrix} = \boxed{\begin{pmatrix} -6 & -4 \\ 2 & 12 \\ 3 & -2 \end{pmatrix}}$$

Matrix Multiplication

Matrices can also be **multiplied** either by a **scalar** or by **another matrix**.

How do I multiply a matrix by a scalar?

- Multiply **each element** in the matrix by the **scalar** value
  - $kA = (ka_{ij})$
- The **resultant** matrix is of the **same order** as the original matrix
- Multiplication by a **negative** scalar changes the **sign** of each element in the matrix

How do I multiply a matrix by another matrix?

- To multiply a matrix by another matrix, the **number of columns** in the **first** matrix must be **equal** to the **number of rows** in the **second** matrix
- If the order of the **first** matrix is $m \times n$ and the order of the **second** matrix is $n \times p$, then the order of the **resultant** matrix will be $m \times p$
- The product of two matrices is found by multiplying the corresponding elements in the **row** of the **first** matrix with the corresponding elements in the **column** of the **second** matrix and finding the **sum** to place in the resultant matrix
  - E.g. If $\boldsymbol{A} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} g & h \\ i & j \\ k & l \end{bmatrix}$
    - then $\boldsymbol{AB} = \begin{bmatrix} (ag + bi + ck) & (ah + bj + cl) \\ (dg + ei + fk) & (dh + ej + fl) \end{bmatrix}$
    - then $\boldsymbol{BA} = \begin{bmatrix} (ga + hd) & (gb + he) & (gc + hf) \\ (ia + jd) & (ib + je) & (ic + jf) \\ (ka + ld) & (kb + le) & (kc + lf) \end{bmatrix}$

How do I square an expression involving matrices?

- If an expression involving matrices is squared then you are multiplying the expression by itself, so write it out in bracket form first, e.g. $(A + B)^2 = (A + B)(A + B)$
  - remember, the regular rules of algebra do not apply here and you cannot expand these brackets, instead, add together the matrices inside the brackets and then multiply the matrices together

What are the properties of matrix multiplication?

- $\boldsymbol{AB} \neq \boldsymbol{BA}$ (non-commutative)
- $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$ (associative)
- $\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC}$ (distributive)
- $(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC} + \boldsymbol{BC}$ (distributive)
- $\boldsymbol{AI} = \boldsymbol{IA} = \boldsymbol{A}$ (identity law)

- $\boldsymbol{AO} = \boldsymbol{OA} = \boldsymbol{O}$, where $\boldsymbol{O}$ is a zero matrix
- Powers of **square** matrices: $\boldsymbol{A}^2 = \boldsymbol{AA},\ \boldsymbol{A}^3 = \boldsymbol{AAA}$ etc.

Worked example

Consider the matrices $\boldsymbol{A} = \begin{bmatrix} 4 & 2 & -5 \\ -3 & 8 & 1 \\ -1 & -2 & 2 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 5 & 1 \\ -2 & 5 \\ 9 & 7 \end{bmatrix}$.

a) Find $\boldsymbol{AB}$.

$$AB = \begin{pmatrix} 4 & 2 & -5 \\ -3 & 8 & 1 \\ -1 & -2 & 2 \end{pmatrix} \times \begin{pmatrix} 5 & 1 \\ -2 & 5 \\ 9 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} (4\times5 + 2\times-2 + -5\times9) & (4\times1 + 2\times5 + -5\times7) \\ (-3\times5 + 8\times-2 + 1\times9) & (-3\times1 + 8\times5 + 1\times7) \\ (-1\times5 + -2\times-2 + 2\times9) & (-1\times1 + -2\times5 + 2\times7) \end{pmatrix}$$

$$= \begin{pmatrix} (20 - 4 - 45) & (4 + 10 - 35) \\ (-15 - 16 + 9) & (-3 + 40 + 7) \\ (-5 + 4 + 18) & (-1 - 10 + 14) \end{pmatrix}$$

$$\boxed{AB = \begin{pmatrix} -29 & -21 \\ -22 & 44 \\ 17 & 3 \end{pmatrix}}$$

b) Explain why you cannot find $\boldsymbol{BA}$.

BA cannot be found because the number of columns in B is different to the number of rows in A

c) Find $\boldsymbol{A}^2$.

$$A^2 = \begin{pmatrix} 4 & 2 & -5 \\ -3 & 8 & 1 \\ -1 & -2 & 2 \end{pmatrix}^2 = \begin{pmatrix} 4 & 2 & -5 \\ -3 & 8 & 1 \\ -1 & -2 & 2 \end{pmatrix}\begin{pmatrix} 4 & 2 & -5 \\ -3 & 8 & 1 \\ -1 & -2 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} (4\times4 + 2\times-3 + -5\times-1) & (4\times2 + 2\times8 + -5\times-2) & (4\times-5 + 2\times1 + -5\times2) \\ (-3\times4 + 8\times-3 + 1\times-1) & (-3\times2 + 8\times8 + 1\times-2) & (-3\times-5 + 8\times1 + 1\times2) \\ (-1\times4 + -2\times-3 + 2\times-1) & (-1\times2 + -2\times8 + 2\times-2) & (-1\times-5 + -2\times1 + 2\times2) \end{pmatrix}$$

$$\boxed{A^2 = \begin{pmatrix} 15 & 34 & -28 \\ -37 & 56 & 25 \\ 0 & -22 & 7 \end{pmatrix}}$$

1.7.3 Determinants & Inverses

Determinants

What is a determinant?

- The **determinant** is a **numerical value** (positive or negative) calculated from the elements in a matrix and is used to find the **inverse** of a matrix
- You can only find the determinant of a **square** matrix
- The method for finding the determinant of a 2×2 matrix is given in your **formula booklet**:

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det \boldsymbol{A} = |\boldsymbol{A}| = ad - bc$$

- You only need to be able to find the determinant of a 2×2 matrix **by hand**
  - For larger $n \times n$ matrices you are expected to **use your GDC**
- The determinant of an **identity matrix** is $\det(\boldsymbol{I}) = 1$
- The determinant of a **zero matrix** is $\det(\boldsymbol{O}) = 0$
- When finding the determinant of a **multiple** of a matrix or the **product** of two matrices:
  - $\det(k\boldsymbol{A}) = k^2 \det(\boldsymbol{A})$ (for a 2×2 matrix)
  - $\det(\boldsymbol{AB}) = \det(\boldsymbol{A}) \times \det(\boldsymbol{B})$

Worked example

Consider the matrix $\boldsymbol{A} = \begin{pmatrix} 3 & -6 \\ p & 7 \end{pmatrix}$, where $p \in \mathbb{R}$ is a constant.

a) Given that $\det \boldsymbol{A} = -3$, find the value of p.

Determinant of a 2×2 matrix	$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det \boldsymbol{A} = \lvert \boldsymbol{A} \rvert = ad - bc$

$\det A = 3 \times 7 - -6 \times p = 21 + 6p$

So, $-3 = 21 + 6p$

$-24 = 6p$

$\boxed{p = -4}$

b) Find the determinant of $4\boldsymbol{A}$.

$\det(4A) = 4^2 \times -3 = \boxed{-48}$

Inverse Matrices

How do I find the inverse of a matrix?

- The determinant can be used to find out if a matrix is invertible or not:
  - If $\det \boldsymbol{A} \neq 0$, then $\boldsymbol{A}$ is **invertible**
  - If $\det \boldsymbol{A} = 0$, then $\boldsymbol{A}$ is singular and does **not** have an inverse
- The method for finding the inverse of a 2×2 matrix is given in your **formula booklet**:

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \ ad \neq bc$$

- You only need to be able to find the inverse of a 2×2 matrix **by hand**
  - For larger $n \times n$ matrices you are expected to **use your GDC**
- The inverse of a square matrix $\boldsymbol{A}$ is the matrix $\boldsymbol{A}^{-1}$ such that the product of these matrices is an **identity** matrix, $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$
  - As a result of this property:
    - $\boldsymbol{AB} = \boldsymbol{C} \Rightarrow \boldsymbol{B} = \boldsymbol{A}^{-1}\boldsymbol{C}$ (pre-multiplying by $\boldsymbol{A}^{-1}$)
    - $\boldsymbol{BA} = \boldsymbol{C} \Rightarrow \boldsymbol{B} = \boldsymbol{C}\boldsymbol{A}^{-1}$ (post-multiplying by $\boldsymbol{A}^{-1}$)

Worked example

Consider the matrices $\boldsymbol{P}=\begin{pmatrix}4 & -2\\ 8 & 2\end{pmatrix}$, $\boldsymbol{Q}=\begin{pmatrix}k & 6\\ -5 & 3\end{pmatrix}$ and $\boldsymbol{R}=\begin{pmatrix}18 & 18\\ 6 & 54\end{pmatrix}$, where k is a constant.

a) Find $\boldsymbol{P}^{-1}$.

Determinant of a 2×2 matrix	$\boldsymbol{A}=\begin{pmatrix}a & b\\ c & d\end{pmatrix} \Rightarrow \det \boldsymbol{A}=\lvert \boldsymbol{A}\rvert = ad-bc$

Inverse of a 2×2 matrix	$\boldsymbol{A}=\begin{pmatrix}a & b\\ c & d\end{pmatrix} \Rightarrow \boldsymbol{A}^{-1}=\frac{1}{\det \boldsymbol{A}}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix},\ ad\neq bc$

$$P^{-1}=\frac{1}{4\times2-(-2)\times8}\begin{pmatrix}2 & 2\\ -8 & 4\end{pmatrix}$$

$$=\frac{1}{24}\begin{pmatrix}2 & 2\\ -8 & 4\end{pmatrix}$$

$$\boxed{P^{-1}=\begin{pmatrix}\frac{1}{12} & \frac{1}{12}\\ -\frac{1}{3} & \frac{1}{6}\end{pmatrix}}$$

b) Given that $\boldsymbol{PQ}=\boldsymbol{R}$ find the value of k.

$$PQ=R \Rightarrow Q=P^{-1}R$$

$$\begin{pmatrix}k & 6\\ -5 & 3\end{pmatrix}=\begin{pmatrix}\frac{1}{12} & \frac{1}{12}\\ -\frac{1}{3} & \frac{1}{6}\end{pmatrix}\begin{pmatrix}18 & 18\\ 6 & 54\end{pmatrix}$$

$$\begin{pmatrix}k & 6\\ -5 & 3\end{pmatrix}=\begin{pmatrix}\left(\frac{1}{12}\times18+\frac{1}{12}\times6\right) & \left(\frac{1}{12}\times18+\frac{1}{12}\times54\right)\\ \left(-\frac{1}{3}\times18+\frac{1}{6}\times6\right) & \left(-\frac{1}{3}\times18+\frac{1}{6}\times54\right)\end{pmatrix}$$

$$\begin{pmatrix}k & 6\\ -5 & 3\end{pmatrix}=\begin{pmatrix}2 & 6\\ -5 & 3\end{pmatrix}$$

$$\boxed{k=2}$$

1.7.4 Solving Systems of Linear Equations with Matrices

Solving Systems of Linear Equations with Matrices

Matrices are used in a huge variety of applications within engineering, computing and business. They are particularly useful for encrypting data and forecasting from given data. Using matrices allows for much larger and more complex systems of linear equations to be solved easily.

How do you set up a system of linear equations using matrices?

- A linear equation can be written in the form $\boldsymbol{A}x = b$, where $\boldsymbol{A}$ is the matrix of **coefficients**
- Note that for a system of linear equations to have a **unique** solution, the matrix of coefficients must be **invertible** and therefore must be a **square** matrix
  - In exams, only invertible matrices will be given (except when solving for eigenvectors)
- You should be able to use matrices to solve a system of up to **two** linear equations both **with and without** your GDC
- You should be able to use a mixture of matrices and technology to solve a system of up to **three** linear equations

How do you solve a system of linear equations with matrices?

- STEP 1

  Write the information in a matrix equation, e.g. for a system of three linear equations $\boldsymbol{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{B}$,

  where the entries into matrix $\boldsymbol{A}$ are the coefficients of x, y and z and matrix $\boldsymbol{B}$ is a column matrix
- STEP 2

  Re-write the equation using the inverse of $\boldsymbol{A}$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{A}^{-1}\boldsymbol{B}$
- STEP 3

  Evaluate the right-hand side to find the values of the unknown variables x, y and z

Worked example

a) Write the system of equations

$$\begin{cases} x+3y-z=-3 \\ 2x+2y+z=2 \\ 3x-y+2z=1 \end{cases}$$

in matrix form.

$$\boxed{\begin{pmatrix} 1 & 3 & -1 \\ 2 & 2 & 1 \\ 3 & -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}}$$

b) Hence solve the simultaneous linear equations.

Re-write the equation in part a) using the inverse matrix

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{10} & \frac{1}{2} & -\frac{3}{10} \\ -\frac{4}{5} & 1 & -\frac{2}{5} \end{pmatrix}\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$$

Use your GDC to find A^{-1}
if it is larger than a 2×2 matrix

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$$

$$\boxed{\begin{aligned} x &= -2 \\ y &= 1 \\ z &= 4 \end{aligned}}$$