



1.7 Matrices

Contents

- ✤ 1.7.1 Introduction to Matrices
- ✤ 1.7.2 Operations with Matrices
- ✤ 1.7.3 Determinants & Inverses
- ✤ 1.7.4 Solving Systems of Linear Equations with Matrices



1.7.1 Introduction to Matrices

Introduction to Matrices

Matrices are a useful way to represent and manipulate data in order to model situations. The elements in a matrix can represent data, equations or systems and have many real-life applications.

What are matrices?

- A matrix is a **rectangular array** of **elements** (numerical or algebraic) that are arranged in **rows** and **columns**
- The order of a matrix is defined by the number of rows and columns that it has
 - The order of a matrix with m rows and n columns is m imes n
- A matrix **A** can be defined by $\mathbf{A} = (a_{ij})$ where i = 1, 2, 3, ..., m and j = 1, 2, 3, ..., n and a_{ij}

refers to the element in row I, column J

Number of columns, n = 3

$$\mathbf{A} = (a_{i,j}) = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \end{pmatrix}$$
 Number of rows, $m = 2$

What type of matrices are there?

- A column matrix (or column vector) is a matrix with a single column, n = 1
- A row matrix is a matrix with a single row, m = 1
- A square matrix is one in which the number of rows is equal to the number of columns, m = n
- Two matrices are equal when they are of the same order and their corresponding elements are equal,
 i.e. a_{ii} = b_{ii} for all elements
- A zero matrix, \boldsymbol{O} , is a matrix in which all the elements are 0, e.g. $\boldsymbol{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
- The identity matrix, $m{I}$, is a square matrix in which all elements along the leading diagonal are 1 and the

rest are 0, e.g. $\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$







1.7.2 Operations with Matrices

Matrix Addition & Subtraction

Just as with ordinary numbers, **matrices** can be **added** together and **subtracted** from one another, provided that they meet certain conditions.

How is addition and subtraction performed with matrices?

- Two matrices of the **same order** can be added or subtracted
- Only corresponding elements of the two matrices are added or subtracted

•
$$A \pm B = (a_{ij}) \pm (b_{ij}) = (a_{ij} \pm b_{ij})$$

• The **resultant** matrix is of the **same order** as the original matrices being added or subtracted **What are the properties of matrix addition and subtraction?**

- A + B = B + A(commutative)
- A + (B + C) = (A + B) + C (associative)
- A + O = A
- $\bullet \quad O-A=-A$
- $\bullet A B = A + (-B)$



Worked example
Consider the matrices
$$A = \begin{pmatrix} -4 & 2 \\ 7 & 3 \\ 1 & -5 \end{pmatrix}, B = \begin{pmatrix} 2 & 6 \\ 5 & -9 \\ -2 & -3 \end{pmatrix}$$
.
a) Find $A + B$.
 $A + 6 = \begin{pmatrix} -4 & 2 \\ 7 & 3 \\ 1 & -5 \end{pmatrix} + \begin{pmatrix} 2 & 6 \\ 5 & -9 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} -2 & 8 \\ 12 & -6 \\ -1 & -8 \end{pmatrix}$
b) Find $A - B$.
 $A - 6 = \begin{pmatrix} -4 & 2 \\ 7 & 3 \\ 1 & -5 \end{pmatrix} - \begin{pmatrix} 2 & 6 \\ 5 & -9 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} -6 & -4 \\ 2 & 12 \\ 3 & -2 \end{pmatrix}$

For more help, please visit <u>www.exampaperspractice.co.uk</u>



Matrix Multiplication

Matrices can also be **multiplied** either by a **scalar** or by **another matrix**.

How do I multiply a matrix by a scalar?

• Multiply each element in the matrix by the scalar value

$$k\mathbf{A} = (ka_{ii})$$

- The resultant matrix is of the same order as the original matrix
- Multiplication by a **negative** scalar changes the **sign** of each element in the matrix

How do I multiply a matrix by another matrix?

- To multiply a matrix by another matrix, the **number of columns** in the **first** matrix must be **equal** to the **number of rows** in the **second** matrix
- If the order of the first matrix is m × n and the order of the second matrix is n × p, then the order of the resultant matrix will be m × p
- The product of two matrices is found by multiplying the corresponding elements in the row of the first matrix with the corresponding elements in the column of the second matrix and finding the sum to place in the resultant matrix

• E.g. If
$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, \mathbf{B} = \begin{bmatrix} g & h \\ i & j \\ i & j \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} (ag + bi + ck) & (ah + bj + cl) \\ (dg + ei + fk) & (dh + ej + fl) \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} (ga + hd) & (gb + he) & (gc + hf) \\ (ia + jd) & (ib + je) & (ic + jf) \end{bmatrix}$$

$$(ka+ld) (kb+le) (kc+lf)$$

How do I square an expression involving matrices?

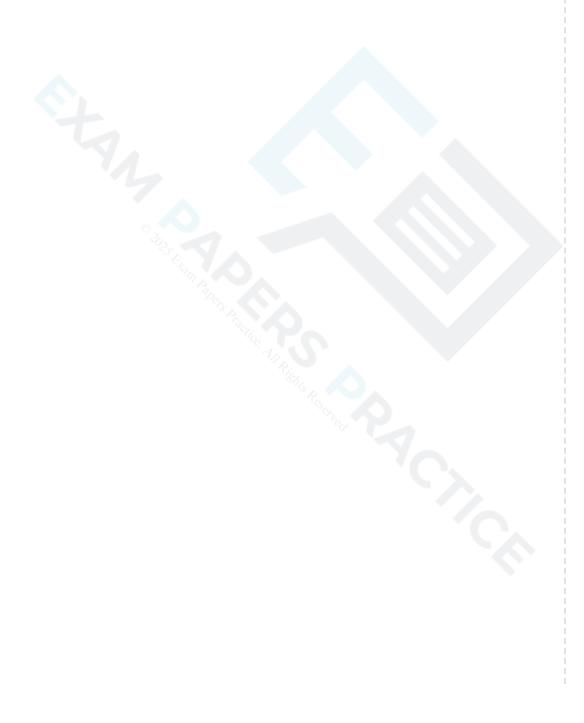
- If an expression involving matrices is squared then you are multiplying the expression by itself, so write it out in bracket form first, e.g. $(A + B)^2 = (A + B)(A + B)$
 - remember, the regular rules of algebra do not apply here and you cannot expand these brackets, instead, add together the matrices inside the brackets and then multiply the matrices together

What are the properties of matrix multiplication?

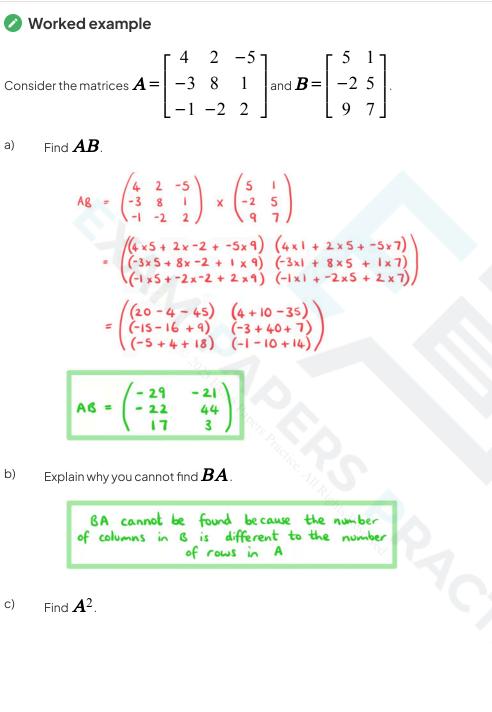
- $AB \neq BA$ (non-commutative)
- A(BC) = (AB)C (associative)
- A(B+C) = AB + AC (distributive)
- (A+B)C = AC + BC (distributive)
- AI = IA = A (identity law)



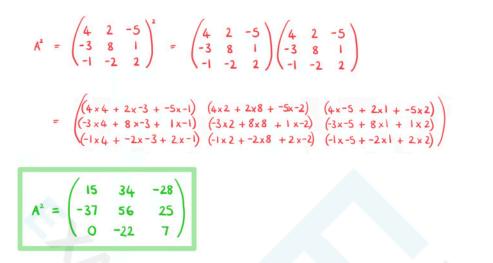
- AO = OA = O, where O is a zero matrix
- Powers of square matrices: $A^2 = AA$, $A^3 = AAA$ etc.













1.7.3 Determinants & Inverses

Determinants

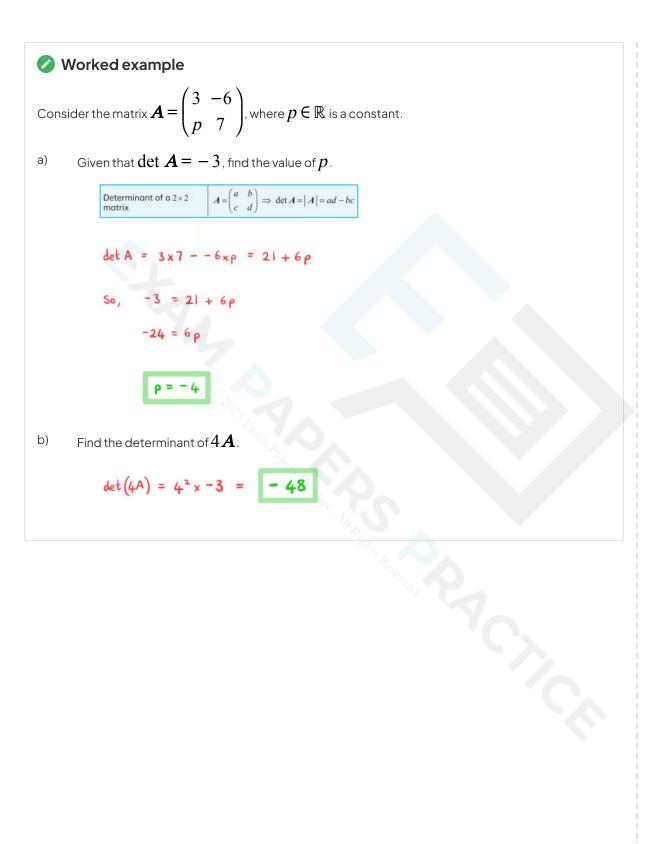
What is a determinant?

- The **determinant** is a **numerical value** (positive or negative) calculated from the elements in a matrix and is used to find the **inverse** of a matrix
- You can only find the determinant of a **square** matrix
- The method for finding the determinant of a 2×2 matrix is given in your formula booklet:

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det \mathbf{A} = |\mathbf{A}| = ad - bc$$

- You only need to be able to find the determinant of a 2×2 matrix by hand
 - For larger $n \times n$ matrices you are expected to use your GDC
- The determinant of an identity matrix is $\det(I) = 1$
- The determinant of a zero matrix is det(O) = 0
- When finding the determinant of a **multiple** of a matrix or the **product** of two matrices:
 - det $(k\mathbf{A}) = k^2 \det(\mathbf{A})$ (for a 2 × 2 matrix)
 - $\det(\mathbf{AB}) = \det(\mathbf{A}) \times \det(\mathbf{B})$







Inverse Matrices

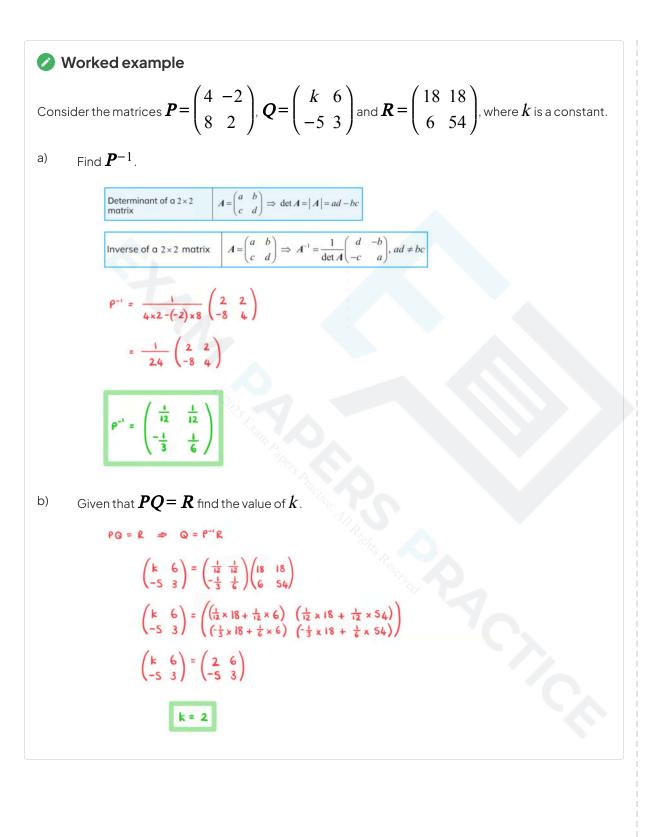
How do I find the inverse of a matrix?

- The determinant can be used to find out if a matrix is invertible or not:
 - If $\det A \neq 0$, then A is invertible
 - If $\det A = 0$, then A is singular and does **not** have an inverse
- The method for finding the inverse of a 2×2 matrix is given in your **formula booklet**:

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \ ad \neq bc$$

- You only need to be able to find the inverse of a 2×2 matrix by hand
 - For larger $n \times n$ matrices you are expected to use your GDC
- The inverse of a square matrix A is the matrix A^{-1} such that the product of these matrices is an identity matrix, $AA^{-1} = A^{-1}A = I$
 - As a result of this property:
 - $AB = C \Rightarrow B = A^{-1}C$ (pre-multiplying by A^{-1})
 - $BA = C \Rightarrow B = CA^{-1}$ (post-multiplying by A^{-1})







1.7.4 Solving Systems of Linear Equations with Matrices

Solving Systems of Linear Equations with Matrices

Matrices are used in a huge variety of applications within engineering, computing and business. They are particularly useful for encrypting data and forecasting from given data. Using matrices allows for much larger and more complex systems of linear equations to be solved easily.

How do you set up a system of linear equations using matrices?

- A linear equation can be written in the form Ax = b, where A is the matrix of **coefficients**
- Note that for a system of linear equations to have a unique solution, the matrix of coefficients must be invertible and therefore must be a square matrix
 - In exams, only invertible matrices will be given (except when solving for eigenvectors)
- You should be able to use matrices to solve a system of up to **two** linear equations both with and without your GDC
- You should be able to use a mixture of matrices and technology to solve a system of up to three linear equations

How do you solve a system of linear equations with matrices?

STEP1

Write the information in a matrix equation, e.g. for a system of three linear equations $oldsymbol{A}$

where the entries into matrix \boldsymbol{A} are the coefficients of X, Y and Z and matrix \boldsymbol{B} is a column matrix

STEP 2

Re-write the equation using the inverse of A, $\begin{bmatrix} A \\ y \\ C \end{bmatrix} = A^{-1}B$

STEP 3

Evaluate the right-hand side to find the values of the unknown variables X, Y and Z



