DP IB Maths: AI HL

1.6 Further Complex Numbers

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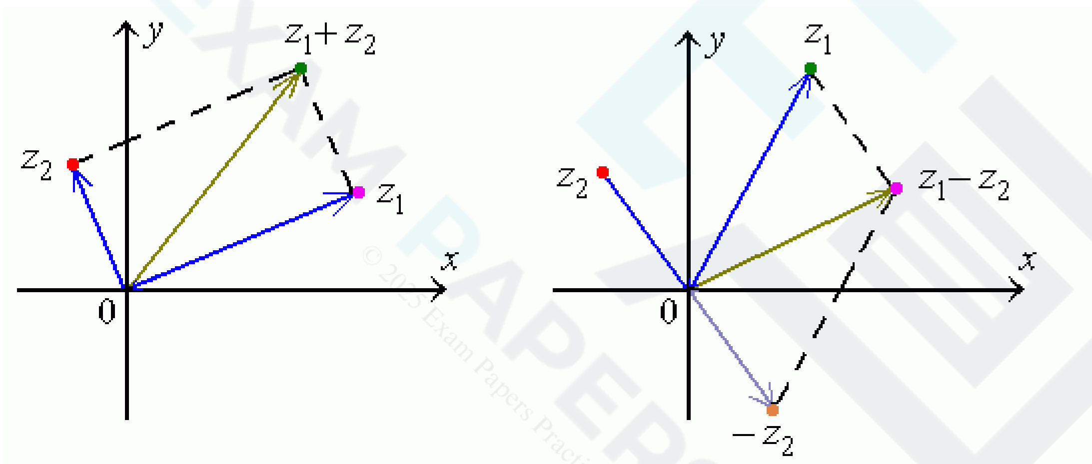
1.6.1 Geometry of Complex Numbers

Geometry of Complex Addition & Subtraction

What does addition look like on an Argand diagram?

- In Cartesian form two complex numbers are added by adding the real and imaginary parts
- When plotted on an Argand diagram the complex number $z_1 + z_2$ is the longer diagonal of the parallelogram with vertices at the origin, z_1, z_2 and $z_1 + z_2$



What does subtraction look like on an Argand diagram?

- In Cartesian form the difference of two complex numbers is found by subtracting the real and imaginary parts
- When plotted on an Argand diagram the complex number $z_1 - z_2$ is the shorter diagonal of the parallelogram with vertices at the origin, z_1, $-z_2$ and $z_1 - z_2$

What are the geometrical representations of complex addition and subtraction?

- Let w be a given complex number with **real part *a*** and **imaginary part *b***
  - $w = a + b\text{i}$
- Let z be any complex number represented on an Argand diagram
- **Adding *w* to *z*** results in z being:
  - **Translated** by vector $\begin{pmatrix} a \\ b \end{pmatrix}$
- **Subtracting *w* from *z*** results in z being:
  - **Translated** by vector $\begin{pmatrix} -a \\ -b \end{pmatrix}$

Worked example

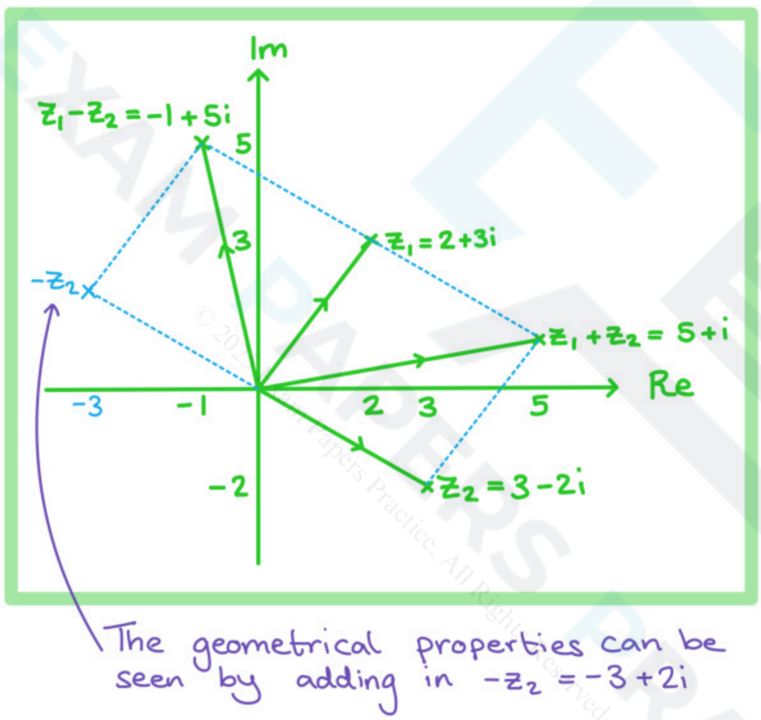
Consider the complex numbers $z_1 = 2 + 3i$ and $z_2 = 3 - 2i$.

On an Argand diagram represent the complex numbers z_1, z_2, $z_1 + z_2$ and $z_1 - z_2$.

First find $z_1 + z_2$ and $z_1 - z_2$:

$z_1 + z_2 = (2 + 3i) + (3 - 2i) = 5 + i$

$z_1 - z_2 = (2 + 3i) - (3 - 2i) = -1 + 5i$



Geometry of Complex Multiplication & Division

What do multiplication and division look like on an Argand diagram?

- The geometrical effect of multiplying a complex number by a real number, a, will be an enlargement of the vector by scale factor a
  - For positive values of a the direction of the vector will not change but the distance of the point from the origin will increase by scale factor a
  - For negative values of a the direction of the vector will change and the distance of the point from the origin will increase by scale factor a
- The geometrical effect of dividing a complex number by a real number, a, will be an enlargement of the vector by scale factor $1/a$
  - For positive values of a the direction of the vector will not change but the distance of the point from the origin will increase by scale factor $1/a$
  - For negative values of a the direction of the vector will change and the distance of the point from the origin will increase by scale factor $1/a$
- The geometrical effect of multiplying a complex number by i will be a rotation of the vector 90° counter-clockwise
  - $\mathrm{i}(x + y\mathrm{i}) = -y + x\mathrm{i}$
- The geometrical effect of multiplying a complex number by an imaginary number, ai, will be a rotation 90° counter-clockwise and an enlargement by scale factor a
  - $a\mathrm{i}(x + y\mathrm{i}) = -ay + ax\mathrm{i}$
- The geometrical effect of multiplying or dividing a complex number by a complex number will be an enlargement and a rotation
  - The direction of the vector will change
    - The angle of rotation is the **argument**
  - The distance of the point from the origin will change
    - The scale factor is the **modulus**

What does complex conjugation look like on an Argand diagram?

- The geometrical effect of plotting a **complex conjugate** on an Argand diagram is a reflection in the real axis
  - The **real** part of the complex number will stay the same and the **imaginary** part will change sign

Worked example

Consider the complex number $z = 2 - \text{i}$.

On an Argand diagram represent the complex numbers z, 3z, iz, z^* and zz^*.

First find $3z$, iz and z^*

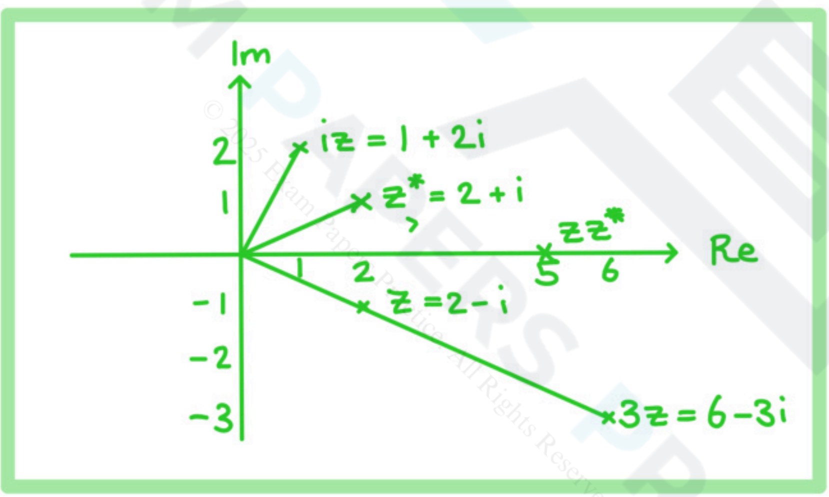
$z = 2 - i$

$3z = 3(2 - i) = 6 - 3i$

$iz = i(2 - i) = 2i - i^2 = 2i - (-1) = 1 + 2i$

$z^* = 2 + i$

$zz^* = (2 - i)(2 + i) = 4 - i^2 = 4 - (-1) = 5$



1.6.2 Forms of Complex Numbers

Modulus-Argument (Polar) Form

How do I write a complex number in modulus-argument (polar) form?

- The **Cartesian form** of a complex number, $z = x + \mathrm{i}y$, is written in terms of its real part, x, and its imaginary part, y
- If we let $r = |z|$ and $\theta = \arg z$, then it is possible to write a complex number in terms of its modulus, r, and its argument, θ, called the **modulus-argument (polar) form**, given by...
  - $z = r(\cos\theta + \mathrm{i}\sin\theta)$
  - This is often written as $z = r\,\mathrm{cis}\,\theta$
  - This is given in the formula book under Modulus-argument (polar) form and exponential (Euler) form
- It is usual to give arguments in the range $-\pi < \theta \leq \pi$ or $0 \leq \theta < 2\pi$
  - Negative arguments should be shown clearly
  - e.g. $z = 2\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right) = 2\,\mathrm{cis}\left(-\frac{\pi}{3}\right)$
    - without simplifying $\cos\left(-\frac{\pi}{3}\right)$ to either $\cos\left(\frac{\pi}{3}\right)$ or $\frac{1}{2}$
- The **complex conjugate** of $r\,\mathrm{cis}\,\theta$ is $r\,\mathrm{cis}\,(-\theta)$
- If a complex number is given in the form $z = r(\cos\theta - \mathrm{i}\sin\theta)$, then it is not in modulus-argument (polar) form due to the minus sign
  - It can be converted by considering transformations of trigonometric functions
    - $-\sin\theta = \sin(-\theta)$ and $\cos\theta = \cos(-\theta)$
  - So $z = r(\cos\theta - \mathrm{i}\sin\theta) = z = r(\cos(-\theta) + \mathrm{i}\sin(-\theta)) = r\,\mathrm{cis}\,(-\theta)$
- To convert from modulus-argument (polar) form back to Cartesian form, evaluate the real and imaginary parts
  - E.g. $z = 2\left(\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right)$ becomes $z = 2\left(\frac{1}{2} + \mathrm{i}\left(-\frac{\sqrt{3}}{2}\right)\right) = 1 - \sqrt{3}\,\mathrm{i}$

How do I multiply complex numbers in modulus-argument (polar) form?

- The main benefit of writing complex numbers in modulus-argument (polar) form is that they multiply and divide very easily
- To **multiply** two complex numbers in modulus-argument (polar) form we **multiply their moduli** and **add their arguments**
  - $|z_1 z_2| = |z_1||z_2|$
  - $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- So if $z_1 = r_1\,\mathrm{cis}\,(\theta_1)$ and $z_2 = r_2\,\mathrm{cis}\,(\theta_2)$
  - $z_1 z_2 = r_1 r_2\,\mathrm{cis}\,(\theta_1 + \theta_2)$

- Sometimes the new argument, $\theta_1 + \theta_2$, does not lie in the range $-\pi < \theta \leq \pi$ (or $0 \leq \theta < 2\pi$ if this is being used)
  - An out-of-range argument can be adjusted by either **adding or subtracting** 2π
  - E.g. If $\theta_1 = \frac{2\pi}{3}$ and $\theta_2 = \frac{\pi}{2}$ then $\theta_1 + \theta_2 = \frac{7\pi}{6}$
  - This is currently not in the range $-\pi < \theta \leq \pi$
  - Subtracting 2π from $\frac{7\pi}{6}$ to give $-\frac{5\pi}{6}$, a new argument is formed
    - This lies in the correct range and represents the same angle on an Argand diagram
- The rules of **multiplying the moduli** and **adding the arguments** can also be applied when…
  - …multiplying three complex numbers together, $z_1 z_2 z_3$, or more
  - …finding powers of a complex number (e.g. z^2 can be written as zz)
- The rules for multiplication can be proved algebraically by multiplying $z_1 = r_1 \text{ cis } (\theta_1)$ by $z_2 = r_2 \text{ cis } (\theta_2)$, expanding the brackets and using compound angle formulae

How do I divide complex numbers in modulus-argument (polar) form?

- To **divide** two complex numbers in modulus-argument (polar) form, we **divide their moduli** and **subtract their arguments**
  - $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
  - $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$
- So if $z_1 = r_1 \text{ cis } (\theta_1)$ and $z_2 = r_2 \text{ cis } (\theta_2)$ then
  - $\frac{z_1}{z_2} = \frac{r_1}{r_2} \text{cis}\left(\theta_1 - \theta_2\right)$
- Sometimes the new argument, $\theta_1 - \theta_2$, can lie out of the range $-\pi < \theta \leq \pi$ (or the range $0 < \theta \leq 2\pi$ if this is being used)
  - You can **add or subtract** 2π to bring out-of-range arguments back in range
- The rules for division can be proved algebraically by dividing $z_1 = r_1 \text{ cis } (\theta_1)$ by $z_2 = r_2 \text{ cis } (\theta_2)$ using **complex division** and the compound angle formulae

Worked example

Let $z_1 = 4\sqrt{2}\ \text{cis}\ \frac{3\pi}{4}$ and $z_2 = \sqrt{8}\left(\cos\left(\frac{\pi}{2}\right) - \text{i}\sin\left(\frac{\pi}{2}\right)\right)$

a) Find $z_1 z_2$, giving your answer in the form $r(\cos\theta + \text{i}\sin\theta)$ where $0 \le \theta < 2\pi$

$z_1 = 4\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)$, $z_2 = \sqrt{8}\left(\cos\left(-\frac{\pi}{2}\right) + \text{i}\sin\left(-\frac{\pi}{2}\right)\right) = 2\sqrt{2}\ \text{cis}\left(-\frac{\pi}{2}\right)$

For $z_1 z_2$, multiply the moduli and add the arguments.

$z_1 z_2 = \left(4\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)\right)\left(2\sqrt{2}\ \text{cis}\left(-\frac{\pi}{2}\right)\right)$

$= (4\sqrt{2})(2\sqrt{2})\ \text{cis}\left(\frac{3\pi}{4} + \left(-\frac{\pi}{2}\right)\right)$

$= 16\ \text{cis}\left(\frac{\pi}{4}\right)$

$$z_1 z_2 = 16\left(\cos\frac{\pi}{4} + \text{i}\sin\frac{\pi}{4}\right)$$

b) Find $\frac{z_1}{z_2}$, giving your answer in the form $r(\cos\theta + \text{i}\sin\theta)$ where $-\pi \le \theta < \pi$

For $\frac{z_1}{z_2}$, divide the moduli and subtract the arguments

$\frac{z_1}{z_2} = \frac{4\sqrt{2}\ \text{cis}\left(\frac{3\pi}{4}\right)}{2\sqrt{2}\ \text{cis}\left(-\frac{\pi}{2}\right)} = \frac{4\sqrt{2}}{2\sqrt{2}}\ \text{cis}\left(\frac{3\pi}{4} - \left(-\frac{\pi}{2}\right)\right)$

$= 2\ \text{cis}\left(\frac{5\pi}{4}\right)$

$= 2\ \text{cis}\left(\frac{5\pi}{4} - 2\pi\right)$

$\frac{5\pi}{4}$ is not in the range $-\pi \le \theta \le \pi$ so subtract 2π to bring it into range

$$\frac{z_1}{z_2} = 2\left(\cos\left(\frac{-3\pi}{4}\right) + \text{i}\sin\left(\frac{-3\pi}{4}\right)\right)$$

Exponential (Euler's) Form

How do we write a complex number in Euler's (exponential) form?

- A complex number can be written in Euler's form as $z = r\mathrm{e}^{\mathrm{i}\theta}$
  - This relates to the modulus-argument (polar) form as $z = r\mathrm{e}^{\mathrm{i}\theta} = r \operatorname{cis} \theta$
  - This shows a clear link between exponential functions and trigonometric functions
  - This is given in the formula booklet under 'Modulus-argument (polar) form and exponential (Euler) form'
- The argument is normally given in the range $0 \le \theta < 2\pi$
  - However in exponential form other arguments can be used and the same convention of adding or subtracting 2π can be applied

How do we multiply and divide complex numbers in Euler's form?

- Euler's form allows for quick and easy multiplication and division of complex numbers
- If $z_1 = r_1\mathrm{e}^{\mathrm{i}\theta_1}$ and $z_2 = r_2\mathrm{e}^{\mathrm{i}\theta_2}$ then
  - $z_1 \times z_2 = r_1 r_2 \mathrm{e}^{\mathrm{i}(\theta_1 + \theta_2)}$
    - Multiply the moduli and add the arguments
  - $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\mathrm{e}^{\mathrm{i}(\theta_1 - \theta_2)}$
    - Divide the moduli and subtract the arguments
- Using these rules makes multiplying and dividing more than two complex numbers much easier than in Cartesian form
- When a complex number is written in Euler's form it is easy to raise that complex number to a power
  - If $z = r\mathrm{e}^{\mathrm{i}\theta}$, $z^2 = r^2\mathrm{e}^{2\mathrm{i}\theta}$ and $z^n = r^n\mathrm{e}^{n\mathrm{i}\theta}$

What are some common numbers in exponential form?

- As $\cos(2\pi) = 1$ and $\sin(2\pi) = 0$ you can write:
  - $1 = \mathrm{e}^{2\pi\mathrm{i}}$
- Using the same idea you can write:
  - $1 = \mathrm{e}^{0} = \mathrm{e}^{2\pi\mathrm{i}} = \mathrm{e}^{4\pi\mathrm{i}} = \mathrm{e}^{6\pi\mathrm{i}} = \mathrm{e}^{2k\pi\mathrm{i}}$
  - where *k* is any integer
- As $\cos(\pi) = -1$ and $\sin(\pi) = 0$ you can write:
  - $\mathrm{e}^{\pi\mathrm{i}} = -1$
  - Or more commonly written as $\mathrm{e}^{\mathrm{i}\pi} + 1 = 0$
    - This is known as Euler's identity and is considered by some mathematicians as the most beautiful equation

- As $\cos\left(\frac{\pi}{2}\right) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$ you can write:
  - $\mathrm{i} = \mathrm{e}^{\frac{\pi}{2}\mathrm{i}}$

Worked example

Consider the complex number $z = 2\mathrm{e}^{\frac{\pi}{3}\mathrm{i}}$. Calculate z^2 giving your answer in the form $r\mathrm{e}^{\mathrm{i}\theta}$.

$$z^2 = \left(2e^{\frac{\pi}{3}i}\right)^2 = \left(2e^{\frac{\pi}{3}i}\right)\left(2e^{\frac{\pi}{3}i}\right) = 4e^{2\left(\frac{\pi}{3}i\right)}$$

multiply the moduli
add the arguments

$$z^2 = 4e^{\frac{2\pi}{3}i}$$

Conversion of Forms

Converting from Cartesian form to modulus-argument (polar) form or exponential (Euler's) form

- To convert from Cartesian form to modulus-argument (polar) form or exponential (Euler) form use
  - $r = |z| = \sqrt{x^2 + y^2}$
- and
  - $\theta = \arg z$

Converting from modulus-argument (polar) form or exponential (Euler's) form to Cartesian form

- To convert from modulus-argument (polar) form to Cartesian form
  - You may need to use your knowledge of trig exact values
  - $a = r\cos\theta$ and $b = r\sin\theta$
  - Write $z = r(\cos\theta + i\sin\theta)$ as $z = r\cos\theta + (r\sin\theta)i$
  - Find the values of the trigonometric ratios $r\sin\theta$ and $r\cos\theta$
  - Rewrite as $z = a + bi$ where
- To convert from exponential (Euler's) form to Cartesian form first rewrite $z = re^{i\theta}$ in the form $z = r\cos\theta + (r\sin\theta)i$ and then follow the steps above

Converting between complex number forms using your GDC

- Your GDC may also be able to convert complex numbers between the various forms
  - TI calculators, for example, have 'Convert to Polar' and 'Convert to Rectangular' (i.e. Cartesian) as options in the 'Complex Number Tools' menu
  - Make sure you are familiar with your GDC and what it can (and cannot) do with complex numbers

Worked example

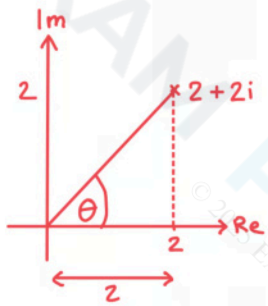
Two complex numbers are given by $z_1 = 2 + 2\mathrm{i}$ and $z_2 = 3\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}$.

a) Write z_1 in the form $r\mathrm{e}^{\mathrm{i}\theta}$.

$z_1 = 2 + 2i$

Find the modulus: $|z_1| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

Draw a sketch to help find the argument:



$\theta = \tan^{-1}\left(\frac{2}{2}\right) = \tan^{-1}(1)$

$= \frac{\pi}{4}$

$r = 2\sqrt{2}, \quad \theta = \frac{\pi}{4}$

$z_1 = 2\sqrt{2}\, e^{\frac{\pi}{4}i}$

b) Write z_2 in the form $r(\cos\theta + \mathrm{i}\sin\theta)$ and then convert it to Cartesian form.

$z_2 = 3e^{\frac{2\pi}{3}i} = 3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= 3\left(-\frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)\right)$

$z_2 = \frac{3}{2}\left(-1 + \sqrt{3}i\right)$

1.6.3 Applications of Complex Numbers

Frequency & Phase of Trig Functions

How are complex numbers and trig functions related?

- A sinusoidal function is of the form $a\sin(bx + c)$
  - a represents the **amplitude**
  - b represents the **period** (also known as frequency)
  - c represents the **phase shift**
    - The function may be written $a\sin(bx + bc) = a\sin b(x + c)$ where the phase shift is represented by bc
    - This will be made clear in the exam
- When written in **modulus-argument** form the **imaginary part** of a complex number relates only to the **sin** part and the **real** part relates to the **cos** part
  - This means that the complex number can be rewritten in Euler's form and relates to the sinusoidal functions as follows:
  - $a\sin(bx + c) = \text{Im}\,(ae^{i(bx + c)})$
  - $a\cos(bx + c) = \text{Re}\,(ae^{i(bx + c)})$
- Complex numbers are particularly useful when working with electrical currents or voltages as these follow sinusoidal wave patterns
  - AC voltages may be given in the form $V = a\sin(bt + c)$ or $V = a\cos(bt + c)$

How are complex numbers used to add two sinusoidal functions?

- Complex numbers can help to add two **sinusoidal** functions if they have the same **frequency** but different **amplitudes** and **phase shifts**
  - e.g. $2\sin(\mathbf{3x} + 1)$ can be added to $3\sin(\mathbf{3x} - 5)$ but **not** $2\sin(\mathbf{5x} + 1)$
- To add $a\sin(bx + c)$ to $d\sin(bx + e)$
  - or $a\cos(bx + c)$ to $d\cos(bx + e)$
- STEP 1: Consider the complex numbers $z_1 = ae^{i(bx + c)}$ and $z_2 = de^{i(bx + e)}$
  - Then $a\sin(bx + c) + d\sin(bx + e) = \text{Im}\,(z_1 + z_2)$
  - Or $a\cos(bx + c) + d\cos(bx + e) = \text{Re}\,(z_1 + z_2)$
- STEP 2: Factorise $z_1 + z_2 = ae^{i(bx + c)} + de^{i(bx + e)} = e^{ibx}(ae^{ci} + de^{ei})$
- STEP 3: Convert $ae^{ci} + de^{ei}$ into a single complex number in exponential form
  - You may need to convert it into Cartesian form first, simplify and then convert back into exponential form
  - Your GDC will be able to do this quickly
- STEP 4: Simplify the whole term and use the rules of indices to collect the powers
- STEP 5: Convert into polar form and take...
  - only the imaginary part for sin
  - or only the real part for cos

Worked example

Two AC voltage sources are connected in a circuit. If $V_1 = 20\sin(30t)$ and $V_2 = 30\sin(30t+5)$ find an expression for the total voltage in the form $V = A\sin(30t+B)$.

$20\sin(30t) + 30\sin(30t+5)$

Frequencies are the same so they can be added

STEP 1: Let $z_1 = 20e^{i(30t)}$ and $z_2 = 30e^{i(30t+5)}$

In polar form the imaginary parts are the sinusoidal functions we want to add.

STEP 2: Find $z_1 + z_2$

$z_1 + z_2 = 20e^{i(30t)} + 30e^{i(30t+5)}$

$= 10e^{30ti}(2 + 3e^{5i})$ ← $2 + (3(\cos(5) + i\sin(5)) = 2.850... - 2.876..i$

STEP 3: Use GDC to find $2 + 3e^{5i}$ in Euler's form

$= 10e^{30ti}(4.05...e^{-0.789...i})$

STEP 4: Use index laws to simplify.

$= 40.5e^{i(30t - 0.789...)}$

STEP 5: Convert to Polar form

$= 40.5(\cos(30t - 0.789...) + i\sin(30t - 0.789...))$

The imaginary part is the solution

$V = 40.5\sin(30t - 0.790)$