



1.6 Binomial Theorem

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1.6.1 Binomial Theorem

Binomial Theorem

What is the Binomial Theorem?

- The binomial theorem (sometimes known as the binomial expansion) gives a method for expanding a two-term expression in a bracket raised to a power
 - A **binomial expression** is in fact any two terms inside the bracket, however in IB the expression will usually be linear
- To expand a bracket with a two-term expression in:
 - First choose the most appropriate parts of the expression to assign to a and b
 - Then use the formula for the binomial theorem:

 $(a+b)^n = a^n + {}^nC_1 a^{n-1}b + \dots + {}^nC_r a^{n-r}b^r + \dots + b^n$

- where ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$
 - See below for more information on ${}^{n}C_{r}$

• You may also see ${}^{n}C_{r}$ written as $\binom{n}{r}$ or ${}_{r}C_{r}$

- You will usually be asked to find the first three or four terms of an expansion
- Look out for whether you should give your answer in **ascending** or **descending** powers of x
 - For **ascending** powers start with the constant term, *aⁿ*
 - For **descending** powers start with the term with x in
 - You may wish to swap a and b over so that you can follow the general formula given in the formula book
- If you are not writing the full expansion you can either
 - show that the sequence continues by putting an ellipsis (...) after your final term
 - or show that the terms you have found are an approximation of the full sequence by using the sign for approximately equals to (≈)

How do I find the coefficient of a single term?

- Most of the time you will be asked to find the coefficient of a term, rather than carry out the whole expansion
- Use the formula for the general term

 ${}^{n}C_{r}a^{n-r}b^{r}$

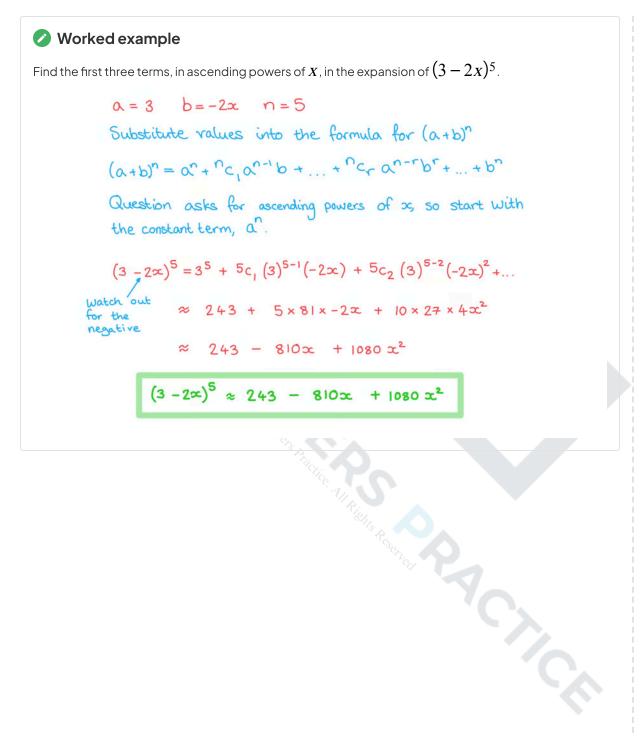
• The question will give you the power of x of the term you are looking for



- Use this to choose which value of r you will need to use in the formula
- This will depend on where the x is in the bracket
- The laws of indices can help you decide which value of *r* to use:
 - For $(a + bx)^n$ to find the coefficient of x^r use $a^{n-r}(bx)^r$
 - For $(a + bx^2)^n$ to find the coefficient of x^r use $a^{n-\frac{r}{2}}(bx^2)^{\frac{r}{2}}$
 - For $\left(a + \frac{b}{x}\right)^n$ look at how the powers will cancel out to decide which value of I to use
 - So for $\left(3x + \frac{2}{x}\right)^8$ to find the coefficient of x^2 use the term with r = 3 and to find the
 - constant term use the term with r = 4
 - There are a lot of variations of this so it is usually easier to see this by inspection of the exponents
- You may also be given the coefficient of a particular term and asked to find an unknown in the brackets
 - Use the laws of indices to choose the correct term and then use the binomial theorem formula to form and solve and equation

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The Binomial Coefficient nCr

What is ${}^{n}C_{r}$?

- If we want to find the number of ways to choose r items out of n different objects we can use the formula for ⁿC_r
 - The formula for *r* combinations of *n* items is ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$
 - This formula is given in the formula booklet along with the formula for the binomial theorem
 - The function ${}^{n}C_{r}$ can be written $\binom{n}{r}$ or ${}^{n}C_{r}$ and is often read as 'n choose r'
 - Make sure you can find and use the button on your GDC

How does ${}^{n}C_{r}$ relate to the binomial theorem?

- The formula ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$ is also known as a **binomial** coefficient
- For a binomial expansion $(a + b)^n$ the coefficients of each term will be nC_0 , nC_1 and so on up to nC

• The coefficient of the r^{th} term will be ${}^{n}C_{r}$

•
$${}^{n}C_{n} = {}^{n}C_{0} = 1$$

- The binomial coefficients are symmetrical, so ${}^{n}C_{r} = {}^{n}C_{n-r}$
 - This can be seen by considering the formula for ${}^{n}C_{r}$

•
$${}^{n}C_{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{r!(n-r)!} = nC_{n-r}$$

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Worked example

Without using a calculator, find the coefficient of the term in x^3 in the expansion of $(1 + x)^9$.

 $n = 9, \quad \alpha = 1, \quad b = \infty$ Substitute values into the formula for the binomial theorem: $(\alpha + b)^{n} = \alpha^{n} + \dots + {}^{n}C_{r} \alpha^{n-r} b^{r} + \dots + b^{n}$ where ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$ $(1 + x)^{9} = \sum_{r=0}^{2} {}^{9}C_{r} (1)^{9-r} (x)^{r} \qquad \text{Coefficient of}$ $(1 + x)^{9} = \sum_{r=0}^{2} {}^{9}C_{r} (1)^{9-r} (x)^{r} \qquad \text{when } r = 3.$ $r = 3 \text{ gives } {}^{9}C_{3} \times (1)^{9-3} (x)^{3}$ Non-calculator, so work out ${}^{n}C_{r}$ separately: $9_{C_{3}} = \frac{9!}{3!(9-3)!} = \frac{9 \times 8 \times 7 \times \cancel{8} \times \cancel{8} \times \cancel{8} \times \cancel{8} \times \cancel{2}}{(3 \times 2)(\cancel{8} \times \cancel{8} \times \cancel{4} \times \cancel{8} \times \cancel{2})}$ $= \frac{9 \times 8 \times 7}{6} = 84$ So the term when r = 3 is $84 \times (1)^{6} \times x^{3}$ $= 84x^{3}$

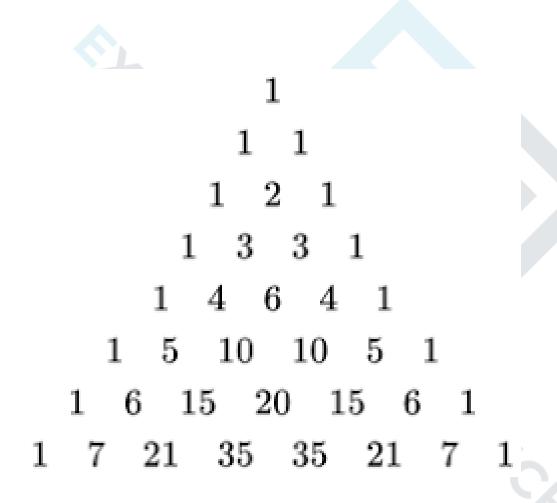
Coefficient of $x^3 = 84$



Pascal's Triangle

What is Pascal's Triangle?

- Pascal's triangle is a way of arranging the binomial coefficients and neatly shows how they are formed
 - Each term is formed by adding the two terms above it
 - The first row has just the number 1
 - Each row begins and ends with a number 1
 - From the third row the terms in between the 1s are the sum of the two terms above it

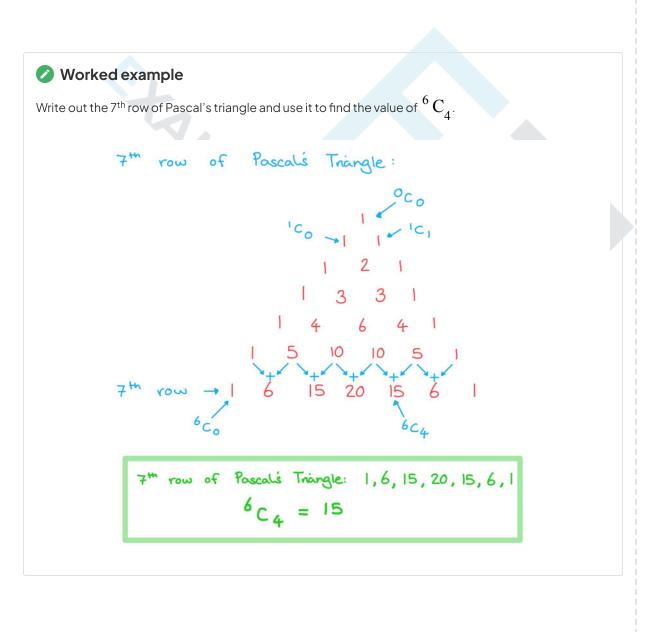


How does Pascal's Triangle relate to the binomial theorem?

- Pascal's triangle is an alternative way of finding the binomial coefficients, ${}^{n}C_{r}$
 - It can be useful for finding for smaller values of n without a calculator



- However for larger values of *n* it is slow and prone to arithmetic errors
- Taking the first row as zero, $\binom{0}{0}C_0 = 1$, each row corresponds to the n^{th} row and the term within that row corresponds to the r^{th} term





1.6.2 Extension of The Binomial Theorem

Binomial Theorem: Fractional & Negative Indices

How do I use the binomial theorem for fractional and negative indices?

The formula given in the formula booklet for the binomial theorem applies to positive integers only

•
$$(a+b)^n = a^n + {}^nC_1a^{n-1}b + \dots + {}^nC_ra^{n-r}b^r + \dots + b^n$$

- where ${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$
- For negative or fractional powers the expression in the brackets must first be changed such that the value for a is 1

•
$$(a+b)^n = a^n \left(1 + \frac{b}{a}\right)^n$$

• $(a+b)^n = a^n \left(1 + n\left(\frac{b}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2 + \dots\right), n \in \mathbb{Q}$

- This is given in the formula booklet
- If a = 1 and b = x the binomial theorem is simplified to

•
$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots, n \in \mathbb{Q}, |x| < 1$$

- This is **not** in the formula booklet, you must remember it or be able to derive it from the formula given
- You need to be able to recognise a negative or fractional power erver
 - The expression may be on the denominator of a fraction

$$\frac{1}{(a+b)^n} = (a+b)^{-n}$$

Or written as a surd

$$\sqrt[m]{(a+b)^m} = (a+b)^{\frac{m}{n}}$$

- For $n \notin \mathbb{N}$ the expansion is infinitely long • You will usually be asked to find the first three terms
- The expansion is only valid for |x| < 1
 - This means -1 < x < 1
 - This is known as the interval of convergence
 - For an expansion $(a + bx)^n$ the interval of convergence would be $-\frac{a}{b} < x < \frac{a}{b}$

How do we use the binomial theorem to estimate a value?



- The binomial expansion can be used to form an approximation for a value raised to a power
- Since |X| < 1 higher powers of x will be very small
 - Usually only the first three or four terms are needed to form an approximation
 - The more terms used the closer the approximation is to the true value
- The following steps may help you use the binomial expansion to approximate a value
 - STEP 1: Compare the value you are approximating to the expression being expanded

e.g.
$$(1 - x)^{\frac{1}{2}} = 0.96^{\frac{1}{2}}$$

- STEP 2: Find the value of x by solving the appropriate equation
 - e.g. 1 x = 0.96

$$x = 0.04$$

• STEP 3: Substitute this value of x into the expansion to find the approximation

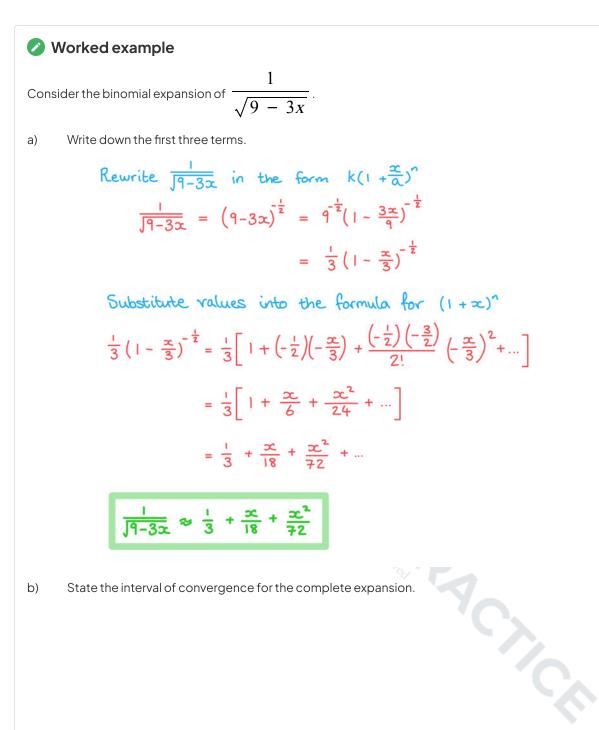
• e.g.
$$1 - \frac{1}{2}(0.04) - \frac{1}{8}(0.04)^2 = 0.9798$$

- Check that the value of x is within the **interval of convergence** for the expression
 - If x is outside the interval of convergence then the approximation may not be valid

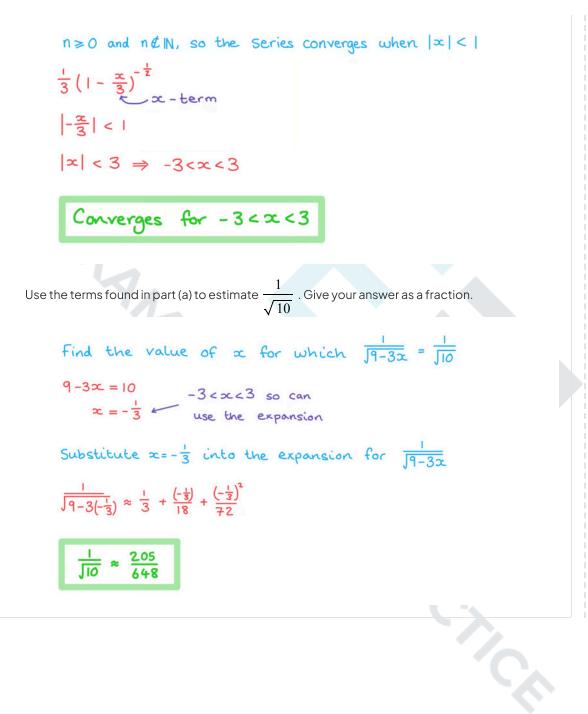
Example Paras

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