1.5 Binomial Theorem

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Course	DP IB Maths
Section	1. Number & Algebra
Topic	1.5 Binomial Theorem
Difficulty	Medium

To be used by all students preparing for DP IB Maths AA SL
Students of other boards may also find this useful

Question 1

$n = 8, \quad a = 2, \quad b = -x$

Substitute values into the formula for the binomial theorem:

$$(a+b)^n = a^n + \ldots + {}^nC_r\, a^{n-r} b^r + \ldots + b^n$$

where $\quad {}^nC_r = \dfrac{n!}{r!(n-r)!}$

$$(2-x)^8 = \sum_{r=0}^{8} {}^8C_r\,(2)^{8-r}\,(-x)^r$$

Coefficient of x^3 occurs when $r=3$.

$r = 3$ gives $\quad {}^8C_3 \times 2^{8-3}\,(-x)^3$

Non-calculator, so work out ${}^nC_r$ separately:

$${}^8C_3 = \frac{8!}{3!(8-3)!} = \frac{8\times7\times6\times\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}}{(3\times2)(\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2})}$$

$$= \frac{8\times7\times\cancel{6}}{\cancel{6}} = 56$$

so the term when $r=3$ is $\quad 56 \times 2^5 \times (-x)^3$

$$= 56 \times -32x^3$$

$$= -1792\,x^3$$

$$\boxed{\text{Coefficient of } x^3 = -1792}$$

Question 2

$a = 3, \quad b = x, \quad n = 4$

Substitute values into the formula for $(a+b)^n$

$$(a+b)^n = a^n + {}^nC_1 a^{n-1} b + \ldots + {}^nC_r a^{n-r} b^r + \ldots + b^n$$

where $\quad {}^nC_r = \dfrac{n!}{r!(n-r)!}$

Question asks for ascending powers of x, so start with the term in x^0.

$$(3+x)^4 = \underbrace{3^4}_{\text{constant term}} + \underbrace{{}^4C_1 (3)^{4-1}(x)^1}_{\text{term in } x} + \underbrace{{}^4C_2 (3)^{4-2}(x)^2}_{\text{term in } x^2} + \ldots$$

$$\approx 81 + \frac{4!}{3!} \times 3^3 \times x + \frac{4!}{2!2!} \times 3^2 \times x^2$$

$$\approx 81 + 4 \times 27x + 6 \times 9x^2$$

$$\approx 81 + 108x + 54x^2$$

$$\boxed{(3+x)^4 \approx 81 + 108x + 54x^2}$$

Question 3

$a = a, \quad b = -x, \quad n = 4$

Substitute values into the formula for $(a+b)^n$

$$(a+b)^n = \sum_{r=0}^{n} {}^nC_r \, a^{n-r} b^r$$

$$(a-x)^4 = \sum_{r=0}^{4} {}^4C_r \, a^{4-r} (-x)^r$$

given the coefficient of the term in x^2, so evaluate the term when $r=2$.

Term in x^2 $= {}^4C_2 (a)^{4-2} (-x)^2 = 96x^2$

$6a^2(-x)^2 = 96x^2$

coefficient of x^2

$6a^2 = 96$

$a^2 = 16$

$a = \pm 4$

It is given in the question that $a > 0 \Rightarrow a = 4$

$$\boxed{a = 4}$$

Question 4

$a = 9 \quad b = -2x \quad n = 5$

Substitute values into the formula for $(a+b)^n$

$$(a+b)^n = a^n + {}^nC_1 a^{n-1} b + \ldots + {}^nC_r a^{n-r} b^r + \ldots + b^n$$

Question asks for ascending powers of x, so start with the constant term, a^n.

$$(9-2x)^5 = 9^5 + 5C_1 (9)^{5-1}(-2x) + 5C_2 (9)^{5-2}(-2x)^2 + \ldots$$

$$\approx 59049 + 5 \times 6561 \times -2x + 10 \times 729 \times 4x^2$$

$$\approx 59049 - 65610x + 29160x^2$$

$$\boxed{(9-2x)^5 \approx 59049 - 65610x + 29160x^2}$$

Question 5

$a = a, \quad b = -2x, \quad n = 5$

Substitute values into the formula for $(a+b)^n$

$$(a+b)^n = \sum_{r=0}^{n} {}^nC_r\, a^{n-r} b^r$$

$$(a-2x)^5 = \sum_{r=0}^{5} {}^5C_r\, a^{5-r}(-2x)^r$$

The coefficients of the terms in x^2 and x^3 are equal, so evaluate the terms when $r=2$ and $r=3$.

In x^2 term, $r = 2$:

$${}^5C_2\, a^{5-2}(-2x)^2$$

$$= 10a^3 \times 4x^2$$

$$= 40a^3x^2$$

coefficient of x^2 term

In x^3 term, $r = 3$:

$${}^5C_3\, a^{5-3}(-2x)^3$$

$$= 10a^2 \times -8x^3$$

$$= -80a^2x^3$$

coefficient of x^3 term

Equating coefficients: $40a^3 = -80a^2$

$$a = \frac{-80}{40} = -2$$

$$\boxed{a = -2}$$

Question 6

$a = 3 \quad b = px \quad n = 6$

$$(3 + px)^6 = \sum_{r=0}^{6} {}^6C_r (3)^{6-r} (px)^r$$

evaluate the terms when $r = 2$ and $r = 4$.

In x^2 term, $r = 2$:

$${}^6C_2 (3)^{6-2} (px)^2$$

$$= 15 \times 81 \times p^2 x^2$$

$$= 1215 p^2 x^2$$

coefficient of x^2

In x^4 term, $r = 4$:

$${}^6C_4 (3)^{6-4} (px)^4$$

$$= 15 \times 9 \times p^4 x^4$$

$$= 135 p^4 x^4$$

coefficient of x^4

coefficient of x^4 = 4(coefficient of x^2)

$$135p^4 = 4(1215p^2)$$

$$135p^4 = 4860p^2$$

$$p^2 = \frac{4860}{135} = 36$$

$$\Rightarrow p = \sqrt{36} = \pm 6$$

$$\boxed{p = 6 \text{ or } -6}$$

Question 7

(a) A binomial expansion has n+1 terms.

$n = 5 \Rightarrow n+1 = 6$

$\boxed{(4ax-3)^5 \text{ has } 6 \text{ terms}}$

(b) $a = 4ax, \quad b = -3, \quad n = 5$

$(4ax-3)^5 = \sum_{r=0}^{5} {}^5C_r\,(4ax)^{5-r}(-3)^r$

we have been given the coefficient of the term in x^4, so evaluate the term when $r = 1$.

Term in $x^4 = -61440x^4$

$\Rightarrow \quad {}^5C_1(4ax)^{5-1}(-3)^1 = -61440x^4$

$5 \times -3 \times 4^4 \times a^4 \times x^4 = -61440x^4$

$-3840a^4 = -61440$

$a^4 = \dfrac{-61440}{-3840} = 16$

$a = \sqrt[4]{16} = \pm 2$

It is given in the question that a is a positive constant.

$\boxed{a = 2}$

Question 8

(a) $a = x^3, \quad b = \frac{4}{x}, \quad n = 4$

$$(a+b)^n = \sum_{r=0}^{n} {}^nC_r \, a^{n-r} b^r$$

$$\left(x^3 + \frac{4}{x}\right)^4 = \sum_{r=0}^{4} {}^4C_r \,(x^3)^{4-r}\left(\frac{4}{x}\right)^r$$

$$= \sum_{r=0}^{4} {}^4C_r \left(\frac{4^r x^{3(4-r)}}{x^r}\right)$$

The numerator has the greater power of x, so start with $r=0$ as the first term

$$\left(x^3 + \frac{4}{x}\right)^4 = {}^4C_0\left(\frac{4^0 x^{3(4-0)}}{x^0}\right) + {}^4C_1\left(\frac{4^1 x^{3(4-1)}}{x^1}\right) + {}^4C_2\left(\frac{4^2 x^{3(4-2)}}{x^2}\right) + \dots$$

$$\approx \frac{1 \times 1x^{12}}{1} + 4 \times \frac{4x^9}{x} + 6 \times \frac{16x^6}{x^2}$$

$$\approx x^{12} + 16x^8 + 96x^4$$

$$\boxed{\left(x^3 + \frac{4}{x}\right)^4 \approx x^{12} + 16x^8 + 96x^4}$$

(b) $\left(x^3 + \frac{4}{x}\right)^4 = \sum_{r=0}^{4} {}^4C_r \, (x^3)^{4-r} \left(\frac{4}{x}\right)^r$

$$= \sum_{r=0}^{4} {}^4C_r \left(\frac{4^r x^{3(4-r)}}{x^r}\right)$$

The constant term is the term in x^0, so we need r such that $3(4-r) - r = 0$

$$3(4-r) - r = 0$$
$$12 - 4r = 0$$
$$r = 3$$

$r = 3$ gives ${}^4C_3 \left(\frac{4^3 \, x^{3(4-3)}}{x^3}\right) = 4 \times 4^3 \times 1$

$= 256$

$\boxed{\text{constant term} = 256}$

Question 9

First expand $(ax+3)^5$, rearranging this to $(3+ax)^5$ makes it easier to spot the correct term to use.

$$(3+ax)^5 = \sum_{r=0}^{5} {}^5C_r\,(3)^{5-r}\,(ax)^r$$

The question gives the term in x^7, so here we are looking for the term in x^4. Evaluate for $r=4$.

The term when $r=4$:

$$(3+ax)^5 = \ldots + {}^5C_4(3)^{5-4}(ax)^4 + \ldots$$

$$= \ldots + 15a^4x^4 + \ldots$$

The coefficient of x^7 in the expansion $x^3(ax+3)^5 = 1215$, therefore:

$$x^3(15a^4)x^4 = 1215x^7$$

$$15a^4 = 1215$$

$$a^4 = 81$$

$$a = \pm\sqrt[4]{81}$$

$$= \pm 3$$

$$\boxed{a = 3 \text{ or } -3}$$