DP IB Maths: AA HL

1.5 Further Proof & Reasoning

Contents

1.5.1 Proof by Induction

Proof by Induction

What is proof by induction?

- **Proof by induction** is a way of proving a **result is true for a set of integers** by showing that if it is **true for one integer then it is true for the next integer**
- It can be thought of as dominoes:
  - All dominoes will fall down if:
    - The first domino falls down
    - Each domino falling down causes the next domino to fall down

What are the steps for proof by induction?

- **STEP 1: The basic step**
  - **Show** the result is true for the **base case**
  - This is **normally *n* = 1 or 0** but it could be any integer
    - For example: To prove $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ is true for all integers $n \geq 1$ you would first need to show it is true for $n = 1$:
    - $\sum_{r=1}^{1} r^2 = \frac{1}{6}(1)((1)+1)(2(1)+1)$
- **STEP 2: The assumption step**
  - **Assume** the result is true for ***n* = *k*** for some integer k
    - For example: Assume $\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$ is true
  - There is nothing to do for this step apart from writing down the assumption
- **STEP 3: The inductive step**
  - **Using the assumption show** the result is true for ***n* = *k* + 1**
  - It can be helpful to simplify LHS & RHS separately and show they are identical
  - The assumption from STEP 2 will be needed at some point
    - For example: $LHS = \sum_{r=1}^{k+1} r^2$ and $RHS = \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+1)$
- **STEP 4: The conclusion step**
  - **State** the result is true
  - **Explain in words** why the result is true
  - It must include:
    - If true for $n = k$ then it is true for $n = k + 1$
    - Since true for $n = 1$ the statement is true for all $n \in \mathbb{Z}$, $n \geq 1$ by mathematical induction

- The sentence will be the same for each proof just change the base case from $n = 1$ if necessary

What type of statements might I be asked to prove by induction?

- **Sums of sequences**
  - If the terms involve factorials then $(k+1)! = (k+1) \times (k!)$ is useful
  - These can be written in the form $\sum_{r=1}^{n} f(r) = g(n)$
  - A useful trick for the inductive step is using $\sum_{r=1}^{k+1} f(r) = f(k+1) + \sum_{r=1}^{k} f(r)$
- **Divisibility** of an expression by an integer
  - These can be written in the form $f(n) = m \times q_n$ where m & q_n are integers
  - A useful trick for the inductive step is using $a^{k+1} = a \times a^k$
- **Complex numbers**
  - You can use proof by induction to prove de Moivre's theorem
- **Derivatives**
  - Such as chain rule, product rule & quotient rule
  - These can be written in the form $f^{(n)}(x) = g(x)$
  - A useful trick for the inductive step is using $f^{(k+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(f^{(k)}(x)\right)$
  - You will have to use the differentiation rules

Worked example

Prove by induction that $\sum_{r=1}^{n} r(r-3) = \frac{1}{3}n(n-4)(n+1)$ for $n \in \mathbb{Z}^+$.

Want to prove $\sum_{r=1}^{n} r(r-3) = \frac{1}{3}n(n-4)(n+1)$

Basic step

Show true for $n=1$

$LHS = \sum_{r=1}^{1} r(r-3) = (1)(1-3) = -2$

$RHS = \frac{1}{3}(1)(1-4)(1+1) = -2$ $\quad \therefore$ LHS = RHS so true for $n=1$

Assumption step

Assume true for $n=k$

Assume $\sum_{r=1}^{k} r(r-3) = \frac{1}{3}k(k-4)(k+1)$

Inductive step

Show true for $n=k+1$

$RHS = \frac{1}{3}(k+1)((k+1)-4)((k+1)+1) = \frac{1}{3}(k+1)(k-3)(k+2)$

$LHS = \sum_{r=1}^{k+1} r(r-3) = (k+1)((k+1)-3) + \sum_{r=1}^{k} r(r-3)$

$= (k+1)(k-2) + \frac{1}{3}k(k-4)(k+1)$ Using assumption

$= \frac{1}{3}(k+1)\left[3(k-2) + k(k-4)\right]$ Factorise $\frac{1}{3}(k+1)$

$= \frac{1}{3}(k+1)\left[k^2 - k - 6\right]$

$= \frac{1}{3}(k+1)(k-3)(k+2)$

$\therefore$ LHS = RHS so true for $n=k+1$

Conclusion step

Explain

If true for $n=k$ then true for $n=k+1$.
Since it is true for $n=1$, the statement is true for all $n \in \mathbb{Z}^+$

$$\sum_{r=1}^{n} r(r-3) = \frac{1}{3}n(n-4)(n+1)$$

1.5.2 Proof by Contradiction

Proof by Contradiction

What is proof by contradiction?

- **Proof by contradiction** is a way of proving a **result is true** by showing that **the negation can not be true**
- It is done by:
  - Assuming the negation (opposite) of the result is true
  - Showing that this then leads to a contradiction

How do I determine the negation of a statement?

- The **negation** of a statement is the **opposite**
  - It is the statement that makes the original statement false
- To negate statements that mention "all", "every", "and" "both":
  - Replace these phrases with "there is at least one", "or" or "there exists" and include the opposite
- To negate statements that mention "there is at least one", "or" or "there exists":
  - Replace these phrases with "all", "every", "and" or "both" and include the opposite
- To negate a statement with "if A occurs then B occurs":
  - Replace with "A occurs and the negation of B occurs"
- Examples include:

Statement	Negation
a is rational	a is irrational
every even number bigger than 2 can be written as the sum of two primes	there exists an even number bigger than 2 which cannot be written as a sum of two primes
n is even and prime	n is not even or n is not prime
there is at least one odd perfect number	all perfect numbers are even
n is a multiple of 5 or a multiple of 3	n is not a multiple of 5 and n is not a multiple of 3
if n^2 is even then n is even	n^2 is even and n is odd

What are the steps for proof by contradiction?

- **STEP 1: Assume the negation** of the statement is **true**
  - You assume it is true but then try to prove your assumption is wrong

- For example: To prove that there is no smallest positive number you start by assuming there is a smallest positive number called *a*
- **STEP 2:** Find **two results which contradict** each other
  - Use algebra to help with this
  - Consider how a contradiction might arise
    - For example: ½*a* is positive and it is smaller than *a* which contradicts that *a* was the smallest positive number
- **STEP 3**: **Explain** why the **original statement is true**
  - In your explanation mention:
    - The **negation can't be true** as it led to a contradiction
    - Therefore the **original statement must be true**

What type of statements might I be asked to prove by contradiction?

- **Irrational numbers**
  - To show $\sqrt[n]{p}$ is irrational where *p* is a prime
    - Assume $\sqrt[n]{p} = \frac{a}{b}$ where *a* & *b* are integers with no common factors and $b \neq 0$
    - Use algebra to show that *p* is a factor of both *a* & *b*
  - To show that $\log_p(q)$ is irrational where *p* & *q* are different primes
    - Assume $\log_p(q) = \frac{a}{b}$ where *a* & *b* are integers with no common factors and $b \neq 0$
    - Use algebra to show $q^b = p^a$
  - To show that *a* or *b* must be irrational if their sum or product is irrational
    - Assume *a* & *b* are rational and write as fractions
    - Show that $a + b$ or ab is rational
- **Prime numbers**
  - To show a polynomial is never prime
    - Assume that it is prime
    - Show there is at least one factor that cannot equal 1
  - To show that there is an infinite number of prime numbers
    - Assume there are *n* primes $p_1, p_2, \ldots, p_n$
    - Show that $p = 1 + p_1 \times p_2 \times \ldots \times p_n$ is a prime that is bigger than the *n* primes
- **Odds and evens**
  - To show that *n* is even if n^2 is even
    - Assume n^2 is even and *n* is odd
    - Show that n^2 is odd
- **Maximum and minimum values**
  - To show that there is no maximum multiple of 3
    - Assume there is a maximum multiple of 3 called *a*
    - Multiply *a* by 3

Worked example

Prove the following statements by contradiction.

a) For any integer n, if n^2 is a multiple of 3 then n is a multiple of 3.

Assume the negation is true for a contradiction.

Assume n^2 is a multiple of 3 and n is not a multiple of 3.

Every integer can be written as one of $3k-1, 3k, 3k+1$ for some $k \in \mathbb{Z}$

As n is not a multiple of 3 then $n = 3k+1$ or $n = 3k-1$ for some $k \in \mathbb{Z}$

If $n = 3k+1$: $n^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$ so not a multiple of 3

If $n = 3k-1$: $n^2 = (3k-1)^2 = 9k^2 - 6k + 1 = 3(3k^2 - 2k) + 1$ so not a multiple of 3

$\therefore n^2$ is not a multiple of 3

This contradicts the statement "n^2 is a multiple of 3".
Therefore the assumption is incorrect.
Therefore if n^2 is a multiple of 3 then n is a multiple of 3.

b) $\sqrt{3}$ is an irrational number.

Assume the negation is true for a contradiction.

Assume $\sqrt{3}$ is rational so can be written $\sqrt{3} = \frac{a}{b}$ where

a and b are integers with no common factors and $b \neq 0$.

Square both sides and rearrange

$3 = \frac{a^2}{b^2} \Rightarrow 3b^2 = a^2 \Rightarrow a^2$ is a multiple of 3 $\Rightarrow a$ is a multiple of 3

Let $a = 3k$ for some $k \in \mathbb{Z}$

$3b^2 = a^2 \Rightarrow 3b^2 = 9k^2 \Rightarrow b^2 = 3k^2 \Rightarrow b^2$ is a multiple of 3

$\therefore$ b and a are multiples of 3

This contradicts the statement "a and b have no common factors".

Therefore the assumption is incorrect.

Therefore $\sqrt{3}$ is irrational.