DP IB Maths: AI HL

1.5 Complex Numbers

Contents

1.5.1 Intro to Complex Numbers

Cartesian Form

What is an imaginary number?

- Up until now, when we have encountered an equation such as $x^2 = -1$ we would have stated that there are "no real solutions"
  - The solutions are $x = \pm\sqrt{-1}$ which are not real numbers
- To solve this issue, mathematicians have defined one of the square roots of negative one as i; an imaginary number
  - $\sqrt{-1} = \mathrm{i}$
  - $\mathrm{i}^2 = -1$
- The square roots of other negative numbers can be found by rewriting them as a multiple of $\sqrt{-1}$
  - using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

What is a complex number?

- Complex numbers have both a real part and an imaginary part
  - For example: $3 + 4\mathrm{i}$
  - The real part is 3 and the imaginary part is 4
    - Note that the imaginary part does not include the 'i'
- Complex numbers are often denoted by z
  - We refer to the real and imaginary parts respectively using $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$
- Two complex numbers are equal if, and only if, both the real and imaginary parts are identical.
  - For example, $3 + 2\mathrm{i}$ and $3 + 3\mathrm{i}$ are **not equal**
- The set of all complex numbers is given the symbol $\mathbb{C}$

What is Cartesian Form?

- There are a number of different forms that complex numbers can be written in
- The form $z = a + b\mathrm{i}$ is known as **Cartesian Form**
  - $a, b \in \mathbb{R}$
  - This is the first form given in the formula booklet
- In general, for $z = a + b\mathrm{i}$
  - $\mathrm{Re}(z) = a$
  - $\mathrm{Im}(z) = b$
- A complex number can be easily represented geometrically when it is in Cartesian Form
- Your GDC may call this **rectangular form**
  - When your GDC is set in rectangular settings it will give answers in Cartesian Form
  - If your GDC is **not** set in a complex mode it will not give any output in complex number form

- Make sure you can find the settings for using complex numbers in Cartesian Form and practice inputting problems
- Cartesian form is the easiest form for adding and subtracting complex numbers

Worked example

a) Solve the equation $x^2 = -9$

$x^2 = -9$

$x = \pm\sqrt{-9}$

Using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ $\quad x = \pm\sqrt{9}\sqrt{-1}$

$x = \pm 3i$

b) Solve the equation $(x+7)^2 = -16$, giving your answers in Cartesian form.

$(x+7)^2 = -16$

$x + 7 = \pm\sqrt{-16}$

$x + 7 = \pm\sqrt{16}\sqrt{-1}$

$x + 7 = \pm 4i$ ← Using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

Rearrange answer into Cartesian form:

$x = -7 \pm 4i$

Complex Addition, Subtraction & Multiplication

How do I add and subtract complex numbers in Cartesian Form?

- Adding and subtracting complex numbers should be done when they are in **Cartesian form**
- When adding and subtracting complex numbers, simplify the real and imaginary parts separately
  - Just like you would when collecting like terms in algebra and surds, or dealing with different components in vectors
  - $(a+b\text{i})+(c+d\text{i})=(a+c)+(b+d)\text{i}$
  - $(a+b\text{i})-(c+d\text{i})=(a-c)+(b-d)\text{i}$

How do I multiply complex numbers in Cartesian Form?

- Complex numbers can be multiplied by a constant in the same way as algebraic expressions:
  - $k(a+b\text{i})=ka+kb\text{i}$
- Multiplying two complex numbers in Cartesian form is done in the same way as multiplying two linear expressions:
  - $(a+b\text{i})(c+d\text{i})=ac+(ad+bc)\text{i}+bd\text{i}^2=\ ac+(ad+bc)\text{i}-bd$
  - This is a complex number with real part $ac-bd$ and imaginary part $ad+bc$
  - The most important thing when multiplying complex numbers is that
    - $\text{i}^2=-1$
- Your GDC will be able to multiply complex numbers in Cartesian form
  - Practise doing this and use it to check your answers
- It is easy to see that multiplying more than two complex numbers together in Cartesian form becomes a lengthy process prone to errors
  - It is easier to multiply complex numbers when they are in different forms and usually it makes sense to convert them from Cartesian form to either Polar form or Euler's form first
- Sometimes when a question describes multiple complex numbers, the notation $z_1,\ z_2,\ \ldots$ is used to represent each complex number

How do I deal with higher powers of i?

- Because $\text{i}^2=-1$ this can lead to some interesting results for higher powers of i
  - $\mathbf{i}^3=\mathbf{i}^2\times\mathbf{i}=\ -\mathbf{i}$
  - $\mathbf{i}^4=(\mathbf{i}^2)^2=(-1)^2=1$
  - $\mathbf{i}^5=(\mathbf{i}^2)^2\ \times\mathbf{i}=\mathbf{i}$
  - $\mathbf{i}^6=(\mathbf{i}^2)^3=(-1)^3=\ -1$
- We can use this same approach of using i^2 to deal with much higher powers
  - $\mathbf{i}^{23}=(\mathbf{i}^2)^{11}\times\mathbf{i}=(-1)^{11}\times\mathbf{i}=\ -\mathbf{i}$
  - Just remember that -1 raised to an even power is 1 and raised to an odd power is -1

Worked example

a) Simplify the expression $2(8-6i)-5(3+4i)$.

Expand the brackets

$2(8-6i)-5(3+4i) = 16 - 12i - 15 - 20i$

Collect the real and imaginary parts

$16 - 15 - 12i - 20i$

Simplify

$1 - 32i$

b) Given two complex numbers $z_1 = 3 + 4i$ and $z_2 = 6 + 7i$, find $z_1 \times z_2$.

Expand the brackets

$(3+4i)(6+7i) = 18 + 21i + 24i + 28i^2$

$= 18 + 21i + 24i + (28)(-1)$ ← Using $i^2 = -1$

Collect the real and imaginary parts

$18 + 21i + 24i - 28 = 18 - 28 + (21 + 24)i$

Simplify

$-10 + 45i$

Complex Conjugation & Division

When **dividing** complex numbers, the **complex conjugate** is used to change the denominator to a real number.

What is a complex conjugate?

- For a given complex number $z = a + bi$, **the complex conjugate of** z **is denoted as** z^*, where $z^* = a - bi$
- If $z = a - bi$ then $z^* = a + bi$
- You will find that:
  - $z + z^*$ is always real because $(a + bi) + (a - bi) = 2a$
    - For example: $(6 + 5i) + (6 - 5i) = 6 + 6 + 5i - 5i = 12$
  - $z - z^*$ is always imaginary because $(a + bi) - (a - bi) = 2bi$
    - For example: $(6 + 5i) - (6 - 5i) = 6 - 6 + 5i - (-5i) = 10i$
  - $z \times z^*$ is always real because $(a + bi)(a - bi) = a^2 + abi - abi - b^2i^2 = a^2 + b^2$ (as $i^2 = -1$)
    - For example: $(6 + 5i)(6 - 5i) = 36 + 30i - 30i - 25i^2 = 36 - 25(-1) = 61$

How do I divide complex numbers?

- To divide two complex numbers:
  - STEP 1: Express the calculation in the form of a fraction
  - STEP 2: Multiply **the top and bottom by the conjugate of the denominator**:
    - $\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \times \frac{c - di}{c - di}$
    - This ensures we are multiplying by 1; so not affecting the overall value
  - STEP 3: Multiply out and simplify your answer
    - This should have a real number as the denominator
  - STEP 4: Write your answer in Cartesian form as two terms, simplifying each term if needed
    - OR convert into the required form if needed
- Your GDC will be able to divide two complex numbers in Cartesian form
  - Practise doing this and use it to check your answers if you can

Worked example

Find the value of $(1+7i) \div (3-i)$.

Rewrite as a fraction: $\dfrac{1+7i}{3-i}$

complex conjugate of $3-i$ is $3+i$

Multiply top and bottom of the fraction by the complex conjugate of the denominator.

$$\frac{1+7i}{3-i} \times \frac{3+i}{3+i} = \frac{(1+7i)(3+i)}{(3-i)(3+i)}$$

$$= \frac{3+i+21i+7i^2}{9+3i-3i-i^2}$$

$i^2 = -1$

the imaginary parts eliminate each other

$$= \frac{3+22i+(-7)}{9-(-1)}$$

Simplify $= \dfrac{-4+22i}{10}$

Write in Cartesian form $= -\dfrac{4}{10} + \dfrac{22}{10}i$

$$\boxed{-\frac{2}{5} + \frac{11}{5}i}$$

Simplify final answer.

1.5.2 Modulus & Argument

Modulus & Argument

How do I find the modulus of a complex number?

- The modulus of a complex number is its **distance** from the origin when plotted on an Argand diagram
- The modulus of z is written $|z|$
- If $z = x + \mathrm{i}y$, then we can use **Pythagoras** to show...
  - $|z| = \sqrt{x^2 + y^2}$
- A modulus is **never negative**

What features should I know about the modulus of a complex number?

- the modulus is related to the complex **conjugate** by...
  - $zz^* = z^*z = |z|^2$
  - This is because $zz^* = (x + \mathrm{i}y)(x - \mathrm{i}y) = x^2 + y^2$
- In general, $|z_1 + z_2| \neq |z_1| + |z_2|$
  - e.g. both $z_1 = 3 + 4\mathrm{i}$ and $z_2 = -3 + 4\mathrm{i}$ have a modulus of 5, but $z_1 + z_2$ simplifies to 8**i** which has a modulus of 8

How do I find the argument of a complex number?

- The argument of a complex number is the **angle** that it makes on an **Argand diagram**
  - The angle must be taken from the **positive real axis**
  - The angle must be in a **counter-clockwise** direction
- Arguments are measured in **radians**
  - They can be given exact in terms of π
- The argument of z is written $\arg z$
- Arguments can be calculated using right-angled **trigonometry**
  - This involves using the tan ratio plus a sketch to decide whether it is positive/negative and acute/obtuse

What features should I know about the argument of a complex number?

- Arguments are usually given in the range $-\pi < \arg z \leq \pi$
  - Negative arguments are for complex numbers in the third and fourth quadrants
  - Occasionally you could be asked to give arguments in the range $0 < \arg z \leq 2\pi$
    - The question will make it clear which range to use
- The argument of zero, $\arg 0$ is undefined (no angle can be drawn)

What are the rules for moduli and arguments under multiplication and division?

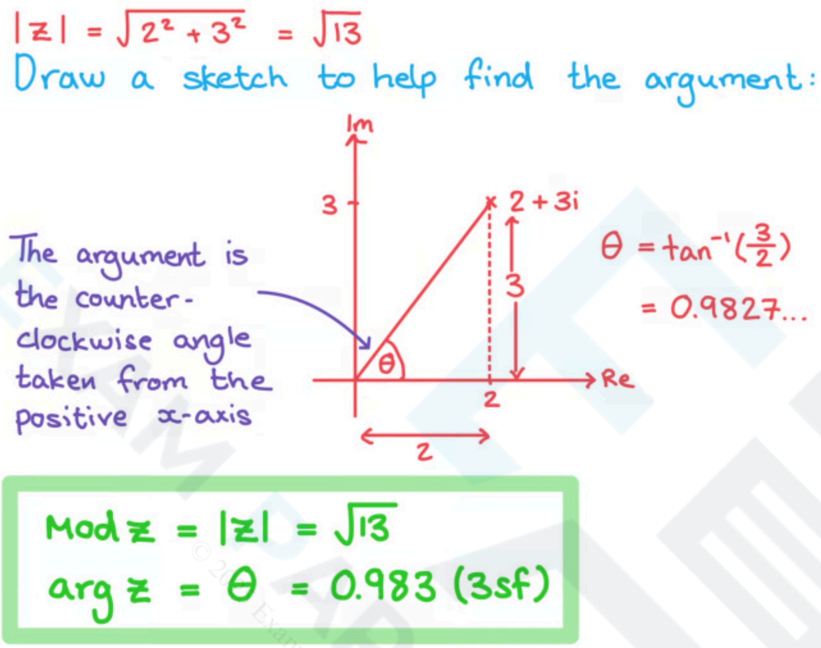
- When two complex numbers, z_1 and z_2, are **multiplied** to give $z_1 z_2$, their **moduli** are also **multiplied**

- $|z_1 z_2| = |z_1||z_2|$

- When two complex numbers, z_1 and z_2, are **divided** to give $\frac{z_1}{z_2}$, their **moduli** are also **divided**
  - $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- When two complex numbers, z_1 and z_2, are **multiplied** to give $z_1 z_2$, their **arguments** are **added**
  - $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
- When two complex numbers, z_1 and z_2, are **divided** to give $\frac{z_1}{z_2}$, their **arguments** are **subtracted**

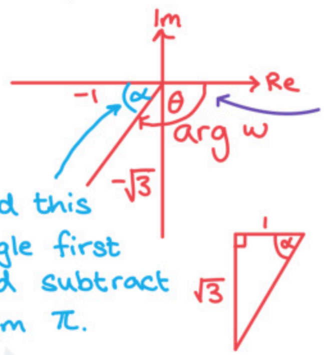
Worked example

a) Find the modulus and argument of $z = 2 + 3\text{i}$



b) Find the modulus and argument of $w = -1 - \sqrt{3}\text{i}$

$|w| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = \sqrt{4}$



If the argument is measured clockwise from the positive x-axis then it will be negative.

Find this angle first and subtract from π.

$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

$\theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\text{Mod } z = |z| = 2$

$\arg z = -\theta = -\frac{2\pi}{3}$

1.5.3 Introduction to Argand Diagrams

Argand Diagrams

What is the complex plane?

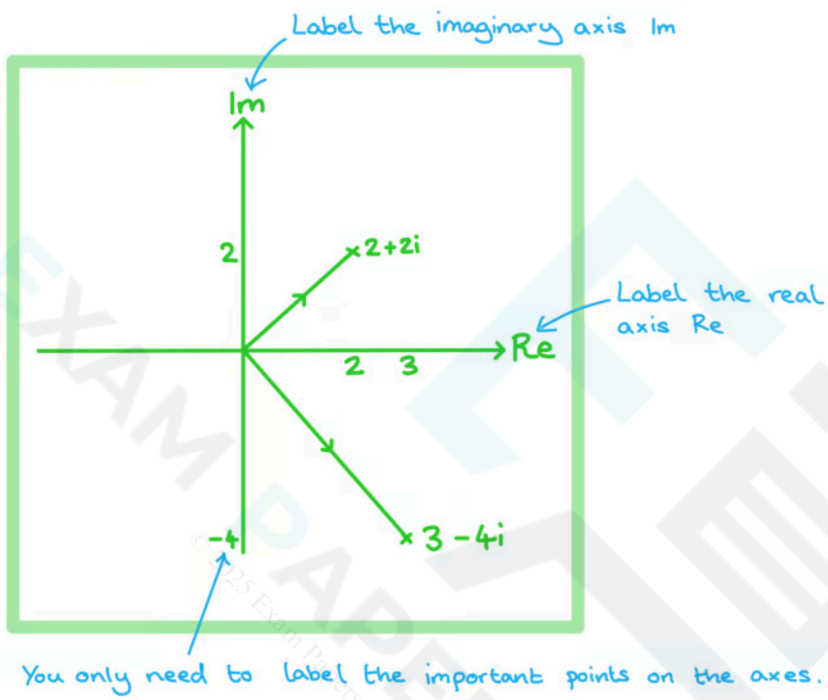
- The complex plane, sometimes also known as the Argand plane, is a two-dimensional plane on which complex numbers can be represented geometrically
- It is similar to a two-dimensional Cartesian coordinate grid
  - The x-axis is known as the **real** axis (Re)
  - The y-axis is known as the **imaginary** axis (Im)
- The complex plane emphasises the fact that a complex number is two dimensional
  - i.e it has two parts, a real and imaginary part
  - Whereas a real number only has one dimension represented on a number line (the x-axis only)

What is an Argand diagram?

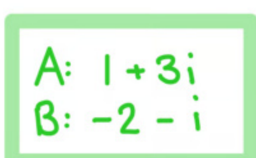
- An Argand diagram is a geometrical representation of complex numbers on a **complex plane**
  - A complex number can be represented as either a point or a vector
- The complex number $x + y\text{i}$ is represented by the point with cartesian coordinate (x, y)
  - The **real** part is represented by the point on the **real** (x-) axis
  - The **imaginary** part is represented by the point on the **imaginary** (y-) axis
- Complex numbers are often represented as **vectors**
  - A line segment is drawn from the origin to the cartesian coordinate point
  - An arrow is added in the direction away from the origin
  - This allows for geometrical representations of complex numbers

Worked example

a) Plot the complex numbers $z_1 = 2 + 2i$ and $z_2 = 3 - 4i$ as points on an Argand diagram.



b) Write down the complex numbers represented by the points A and B on the Argand diagram below.



Complex Roots of Quadratics

What are complex roots?

- A quadratic equation can either have two real roots (zeros), a repeated real root or no real roots
  - This depends on the location of the graph of the quadratic with respect to the x-axis
- If a quadratic equation has no real roots we would previously have stated that it has **no real solutions**
  - The quadratic equation will have a **negative discriminant**
  - This means taking the square root of a negative number
- Complex numbers provide solutions for quadratic equations that have **no real roots**

How do we solve a quadratic equation when it has complex roots?

- If a quadratic equation takes the form $ax^2 + bx + c = 0$ it can be solved by either using the quadratic formula or completing the square
- If a quadratic equation takes the form $ax^2 + b = 0$ it can be solved by rearranging
- The property $i = \sqrt{-1}$ is used
  - $\sqrt{-a} = \sqrt{a \times -1} = \sqrt{a} \times \sqrt{-1}$
- If the coefficients of the quadratic are real then the complex roots will occur in complex conjugate pairs
  - If $\boldsymbol{z = p + qi}$ **(**$\boldsymbol{q \neq 0}$**)** is a root of a quadratic with real coefficients then $\boldsymbol{z^* = p - qi}$ is also a root
- The **real part** of the solutions will have the same value as the x coordinate of the turning point on the graph of the quadratic
- When the coefficients of the quadratic equation are **non-real**, the solutions will **not** be complex conjugates
  - To solve these you can use the quadratic formula

How do we factorise a quadratic equation if it has complex roots?

- If we are given a quadratic equation in the form $az^2 + bz + c = 0$, where a, b, and $c \in \mathbb{R}$, $a \neq 0$ we can use its complex roots to write it in **factorised form**
  - Use the quadratic formula to find the two roots, $z = p + qi$ and $z^* = p - qi$
  - This means that $z - (p + qi)$ and $z - (p - qi)$ must both be factors of the quadratic equation
  - Therefore we can write $az^2 + bz + c = a(z - (p + qi))(z - (p - qi))$
  - This can be rearranged into the form $a(z - p - qi)(z - p + qi)$

Worked example

Solve the quadratic equation $z \ \ - 2z + 5 = 0$ and hence, factorise $z \ \ - 2z + 5$.

Use the quadratic formula or completing the square to find the solutions.

Solutions of a quadratic equation	$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \ a \neq 0$

$a = 1$
$b = -2$
$c = 5$

$$z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm \sqrt{16}\sqrt{-1}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$z_1 = 1 + 2i \qquad z_2 = 1 - 2i$$

If the solutions are $z_1 = 1 + 2i$ and $z_2 = 1 - 2i$ then the factors must be $z - (1 + 2i)$ and $z - (1 - 2i)$

$$z^2 - 2z + 5 = (z - (1 + 2i))(z - (1 - 2i))$$

$$(z - 1 - 2i)(z - 1 + 2i)$$