DP IB Maths: AA SL

1.5 Binomial Theorem

Contents

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Binomial Theorem

What is the Binomial Theorem?

- The **binomial theorem (**sometimes known as the binomial expansion) gives a method for expanding a **two-term** expression in a bracket raised to a power
  - A **binomial expression** is in fact any two terms inside the bracket, however in IB the expression will usually be linear
- To expand a bracket with a two-term expression in:
  - First choose the most appropriate parts of the expression to assign to *a* and *b*
  - Then use the formula for the binomial theorem:

$$(a+b)^n = a^n + {}^nC_1\, a^{n-1} b + \ldots + {}^nC_r\, a^{n-r} b^r + \ldots + b^n$$

  - where ${}^nC_r = \dfrac{n!}{r!(n-r)!}$
    - See below for more information on ${}^nC_r$
    - You may also see ${}^nC_r$ written as $\begin{pmatrix} n \\ r \end{pmatrix}$ or ${}_nC_r$

- You will usually be asked to find the first three or four terms of an expansion
- Look out for whether you should give your answer in **ascending** or **descending** powers of *x*
  - For **ascending** powers start with the constant term, a^n
  - For **descending** powers start with the term with *x* in
    - You may wish to swap *a* and *b* over so that you can follow the general formula given in the formula book
- If you are not writing the full expansion you can either
  - show that the sequence continues by putting an ellipsis (…) after your final term
  - or show that the terms you have found are an approximation of the full sequence by using the sign for approximately equals to (≈)

How do I find the coefficient of a single term?

- Most of the time you will be asked to find the coefficient of a term, rather than carry out the whole expansion
- Use the formula for the general term

$$^nC_r\, a^{n-r}\, b^r$$

- The question will give you the power of *x* of the term you are looking for

- Use this to choose which value of *r* you will need to use in the formula
- This will depend on where the *x* is in the bracket
- The laws of indices can help you decide which value of *r* to use:
    - For $(a + bx)^n$ to find the coefficient of x^r use $a^{n-r}(bx)^r$
    - For $(a + bx^2)^n$ to find the coefficient of x^r use $a^{n-\frac{r}{2}}(bx^2)^{\frac{r}{2}}$
    - For $(a + \frac{b}{x})^n$ look at how the powers will cancel out to decide which value of r to use
    - So for $\left(3x + \frac{2}{x}\right)^8$ to find the coefficient of x^2 use the term with $r = 3$ and to find the constant term use the term with $r = 4$
    - There are a lot of variations of this so it is usually easier to see this by inspection of the exponents
- You may also be given the coefficient of a particular term and asked to find an unknown in the brackets
    - Use the laws of indices to choose the correct term and then use the binomial theorem formula to form and solve and equation

Worked example

Find the first three terms, in ascending powers of x, in the expansion of $(3-2x)^5$.

$a = 3 \quad b = -2x \quad n = 5$

Substitute values into the formula for $(a+b)^n$

$$(a+b)^n = a^n + {}^nC_1 a^{n-1} b + \ldots + {}^nC_r a^{n-r} b^r + \ldots + b^n$$

Question asks for ascending powers of x, so start with the constant term, a^n.

$$(3-2x)^5 = 3^5 + {}^5C_1 (3)^{5-1}(-2x) + {}^5C_2 (3)^{5-2}(-2x)^2 + \ldots$$

Watch out for the negative

$$\approx 243 + 5 \times 81 \times -2x + 10 \times 27 \times 4x^2$$

$$\approx 243 - 810x + 1080x^2$$

$$(3-2x)^5 \approx 243 - 810x + 1080x^2$$

The Binomial Coefficient nCr

What is ${}^{n}C_{r}$?

- If we want to find the number of ways to **choose** *r* items out of *n* different objects we can use the formula for ${}^{n}C_{r}$
  - The formula for *r* **combinations** of *n* items is ${}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$
  - This formula is given in the formula booklet along with the formula for the binomial theorem
  - The function ${}^{n}C_{r}$ can be written $\binom{n}{r}$ or ${}_{n}C_{r}$ and is often read as '*n choose r*'
    - Make sure you can find and use the button on your GDC

How does ${}^{n}C_{r}$ relate to the binomial theorem?

- The formula ${}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$ is also known as a **binomial** coefficient
- For a binomial expansion $(a + b)^n$ the coefficients of each term will be ${}^{n}C_{0}$, ${}^{n}C_{1}$ and so on up to ${}^{n}C_{n}$
  - The coefficient of the r^{th} term will be ${}^{n}C_{r}$
- ${}^{n}C_{n} = {}^{n}C_{0} = 1$
- The binomial coefficients are symmetrical, so ${}^{n}C_{r} = {}^{n}C_{n-r}$
  - This can be seen by considering the formula for ${}^{n}C_{r}$
  - ${}^{n}C_{n-r} = \dfrac{n!}{(n-r)!(n-(n-r))!} = \dfrac{n!}{r!(n-r)!} = nC_{r}$

Worked example

Without using a calculator, find the coefficient of the term in x^3 in the expansion of $(1 + x)^9$.

$n = 9, \quad a = 1, \quad b = x$

Substitute values into the formula for the binomial theorem:

$$(a+b)^n = a^n + \ldots + {}^nC_r a^{n-r} b^r + \ldots + b^n$$

where $\quad {}^nC_r = \dfrac{n!}{r!(n-r)!}$

$$(1+x)^9 = \sum_{r=0}^{9} {}^9C_r (1)^{9-r} (x)^r$$

← Coefficient of x^3 occurs when $r = 3$.

$r = 3$ gives ${}^9C_3 \times (1)^{9-3} (x)^3$

Non-calculator, so work out ${}^nC_r$ separately:

$${}^9C_3 = \frac{9!}{3!(9-3)!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2}{(3 \times 2)(6 \times 5 \times 4 \times 3 \times 2)}$$

$$= \frac{9 \times 8 \times 7}{6} = 84$$

so the term when $r = 3$ is $84 \times (1)^6 \times x^3$

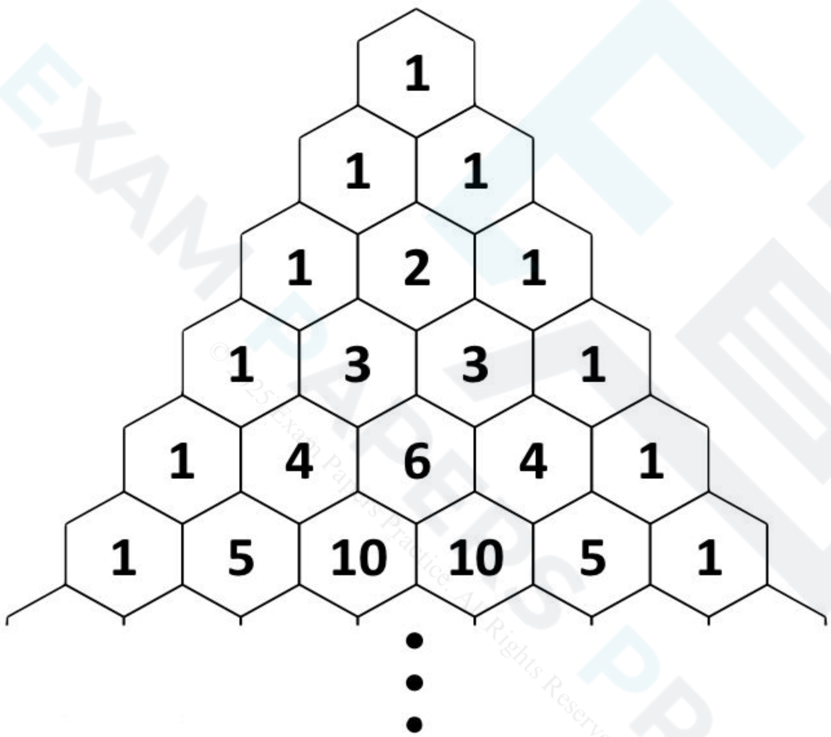
$$= 84x^3$$

Coefficient of x^3 = 84

Pascal's Triangle

What is Pascal's Triangle?

- Pascal's triangle is a way of arranging the binomial coefficients and neatly shows how they are formed
  - Each term is formed by adding the two terms above it
  - The first row has just the number 1
  - Each row begins and ends with a number 1
  - From the third row the terms in between the 1s are the sum of the two terms above it



How does Pascal's Triangle relate to the binomial theorem?

- Pascal's triangle is an alternative way of finding the binomial coefficients, ${}^{n}\mathrm{C}_{r}$
  - It can be useful for finding for smaller values of n without a calculator
  - However for larger values of n it is slow and prone to arithmetic errors
- Taking the first row as zero, $\left({}^{0}\mathrm{C}_{0} = 1\right)$, each row corresponds to the n^{th} row and the term within that row corresponds to the r^{th} term

Worked example

Write out the 7th row of Pascal's triangle and use it to find the value of ${}^6C_4$.