DP IB Maths: AA HL

1.4 Simple Proof & Reasoning

Contents

1.4.1 Proof

Language of Proof

What is proof?

- Proof is a series of logical steps which show a result is **true** for all specified numbers
  - 'Seeing' that a result works for a few numbers is not enough to **show** that it will work for all numbers
  - Proof allows us to show (usually algebraically) that the result will work for **all values**
- You must be familiar with the notation and language of proof
- LHS and RHS are standard abbreviations for left-hand side and right-hand side
- **Integers** are used frequently in the language of proof
  - The set of **integers** is denoted by $\mathbb{Z}$
  - The set of **positive integers** is denoted by $\mathbb{Z}^+$

How do we prove a statement is true for all values?

- Most of the time you will need to use algebra to show that the left-hand side (LHS) is the same as the right-hand side (RHS)
  - You **must not** move terms from one side to the other
  - Start with one side (usually the LHS) and manipulate it to show that it is the same as the other
- A **mathematical identity** is a statement that is true for all values of x (or θ in trigonometry)
  - The symbol $\equiv$ is used to identify an identity
  - If you see this symbol then you can use proof methods to show it is true
- You can complete your proof by stating that RHS = LHS or writing QED

Worked example

Prove that $(2x-2)(x-3)+2(x-1)=2(x-2)(x-1)$.

Work with LHS first:

Expand brackets:

FOIL

LHS: $(2x-2)(x-3)+2(x-1)$

$2x^2-6x-2x+6+2x-2$

Simplify, take care with negatives:

$2x^2-6x+4$

Factorise the 2:

$2(x^2-3x+2)$

Factorise remaining quadratic:

$2(x-2)(x-1)$ = RHS as required.

$$\boxed{(2x-2)(x-3)+2(x-1)=2(x-2)(x-1)}$$

Proof by Deduction

What is proof by deduction?

- A mathematical and logical argument that shows that a result is true

How do we do proof by deduction?

- A proof by deduction question will often involve showing that a result is true for all integers, consecutive integers or even or odd numbers
  - You can begin by letting an integer be n
    - Use conventions for even ($2n$) and odd ($2n - 1$) numbers
- You will need to be familiar with sets of numbers $(\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R})$
  - $\mathbb{N}$ – the set of **natural numbers**
  - $\mathbb{Z}$ – the set of **integers**
  - $\mathbb{Q}$ – the set of **quotients (rational numbers)**
  - $\mathbb{R}$ – the set of **real numbers**

What is proof by exhaustion?

- Proof by exhaustion is a way to show that the desired result works for every allowed value
  - This is a good method when there are only a limited number of cases to check
- Using proof by exhaustion means testing every allowed value not just showing a few examples
  - The allowed values could be specific values
  - They could also be split into cases such as even and odd

Worked example

Prove that the sum of any two consecutive odd numbers is always even.

Let $2n - 1$ be an odd number

must be even

Let two consecutive odd integers be:

$2n-1$, $2n+1$ next odd number

Then their sum is:

$$2n - 1 + 2n + 1 \equiv 4n$$

$$= 2(2n)$$

$\boxed{\text{Any multiple of 2 must be even.}}$

Disproof by Counter Example

What is disproof by counter-example?

- Disproving a result involves finding a value that does **not** work in the result
- That value is called a **counter**-**example**

How do I disprove a result?

- You only need to find **one** value that does not work
- Look out for the set of numbers for which the statement is made, it will often be just integers or natural numbers
- Numbers that have unusual results are often involved
  - It is often a good idea to try the values 0 and 1 first as they often behave in different ways to other numbers
  - The number 2 also behaves differently to other even numbers
    - It is the only even prime number
    - It is the only number that satisfies $n + n = n^n$
  - If it is the set of real numbers consider how rational and irrational numbers behave differently
  - Think about how positive and negative numbers behave differently
    - Particularly when working with inequalities

Worked example

For each of the following statements, show that they are false by giving a counterexample:

a) Given $n \in \mathbb{Z}^+$, if n^2 is a multiple of 4, then n is also a multiple of 4.

$n \in \mathbb{Z}^+ \leftarrow$ set of positive integers only

We are only interested in positive integers so start by trying 1, 2 etc.

$1^2 = 1$ (not a multiple of 4)

$2^2 = 4$ $\quad n^2 = 4$ (multiple of 4)

$n = 2$ (not a multiple of 4)

$$\boxed{\begin{aligned} &\text{Let } n = 2:\ n^2 = 4 \text{ (multiple of 4)} \\ &\qquad\qquad n = 2 \text{ (not a multiple of 4)} \end{aligned}}$$

b) Given $x \in \mathbb{Z}$ then $3x$ is always greater than $2x$.

Your notes

$x \in \mathbb{Z} \leftarrow$ set of integers only

We are interested in both positive and negative integers and zero so consider how each of these groups behave:

Positive integers, e.g. let $x = 1$:

$2x = 2$

$3x = 3 \quad \therefore 3x > 2x$

Zero:

Let $x = 0$

$2x = 0$

$3x = 0 \quad \therefore \quad 3x = 2x$ (this is enough to disprove the result)

Negative integers, e.g. let $x = -1$:

$2x = -2$

$3x = -3 \quad \therefore 2x > 3x$ (Any negative integer can disprove the result)

Let $x = 0$

$2x = 0$

$3x = 0 \quad \therefore \quad 3x \not> 2x$