EXAM PAPERS PRACTICE

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1.4 Simple Proof & Reasoning

IB Maths - Revision Notes

AA HL

1.4.1 Proof

Language of Proof

What is proof?

- Proof is a series of logical steps which show a result is **true** for all specified numbers
  - 'Seeing' that a result works for a few numbers is not enough to **show** that it will work for all numbers
  - Proof allows us to show (usually algebraically) that the result will work for **all values**
- You must be familiar with the notation and language of proof
- LHS and RHS are standard abbreviations for left-hand side and right-hand side
- **Integers** are used frequently in the language of proof
  - The set of **integers** is denoted by $\mathbb{Z}$
  - The set of **positive integers** is denoted by $\mathbb{Z}^+$

How do we prove a statement is true for all values?

- Most of the time you will need to use algebra to show that the left-hand side (LHS) is the same as the right-hand side (RHS)
  - You **must not** move terms from one side to the other
  - Start with one side (usually the LHS) and manipulate it to show that it is the same as the other
- A **mathematical identity** is a statement that is true for all values of x (or θ in trigonometry)
  - The symbol $\equiv$ is used to identify an identity
  - If you see this symbol then you can use proof methods to show it is true
- You can complete your proof by stating that RHS = LHS or writing QED

Exam Tip

- You will need to show each step of your proof clearly and set out your method in a logical manner in the exam
  - Be careful not to skip steps

Worked example

Prove that $(2x-2)(x-3)+2(x-1)=2(x-2)(x-1)$.

Work with LHS first:

Expand brackets:

FOIL

LHS: $(2x-2)(x-3)+2(x-1)$

$2x^2-6x-2x+6+2x-2$

Simplify, take care with negatives:

$2x^2-6x+4$

Factorise the 2:

$2(x^2-3x+2)$

Factorise remaining quadratic:

$2(x-2)(x-1)$ = RHS as required.

$$(2x-2)(x-3)+2(x-1)=2(x-2)(x-1)$$

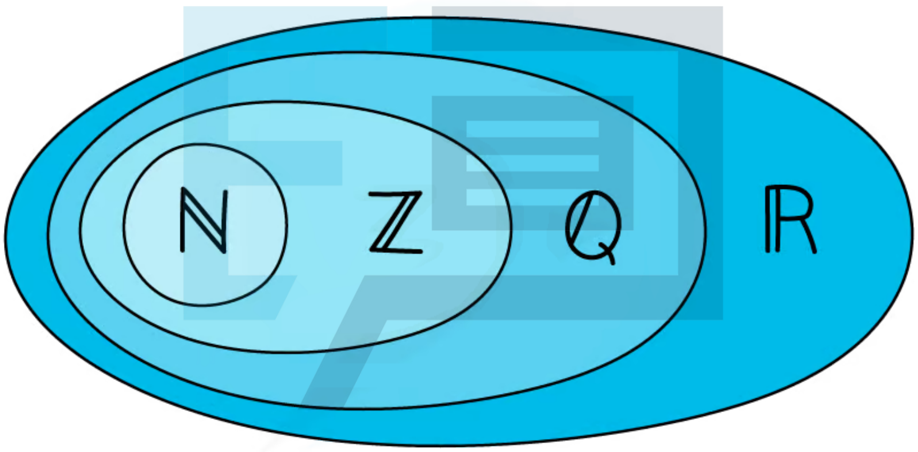
Proof by Deduction

What is proof by deduction?

- A mathematical and logical argument that shows that a result is true

How do we do proof by deduction?

- A proof by deduction question will often involve showing that a result is true for all integers, consecutive integers or even or odd numbers
  - You can begin by letting an integer be *n*
    - Use conventions for even (*2n*) and odd (*2n – 1*) numbers
- You will need to be familiar with sets of numbers $(\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R})$
  - $\mathbb{N}$ – the set of **natural numbers**
  - $\mathbb{Z}$ – the set of **integers**
  - $\mathbb{Q}$ – the set of **quotients (rational numbers)**
  - $\mathbb{R}$ – the set of **real numbers**



What is proof by exhaustion?

- **Proof by exhaustion** is a way to show that the desired result works for every allowed value
  - This is a good method when there are only a limited number of cases to check
- Using proof by exhaustion means testing every allowed value not just showing a few examples
  - The allowed values could be specific values
  - They could also be split into cases such as even and odd

Exam Tip

- Try the result you are proving with a few different values
  - Use a sequence of them (eg 1, 2, 3)
  - Try different types of numbers (positive, negative, zero)
- This may help you see a pattern and spot what is going on

Worked example

Prove that the sum of any two consecutive odd numbers is always even.

Let $2n - 1$ be an odd number

($2n$ must be even)

Let two consecutive odd integers be:

$2n - 1, \; 2n + 1$ (next odd number)

Then their sum is:

$$2n - 1 + 2n + 1 \equiv 4n$$
$$= 2(2n)$$

Any multiple of 2 must be even.

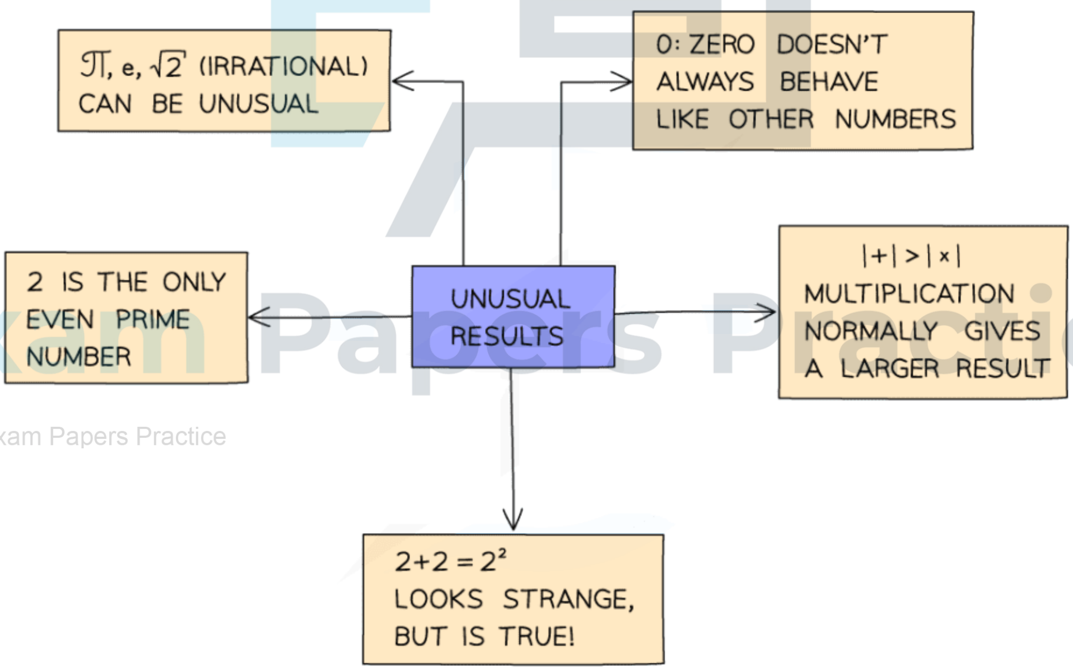
Disproof by Counter Example

What is disproof by counter-example?

- Disproving a result involves finding a value that does **not** work in the result
- That value is called a **counter-example**

How do I disprove a result?

- You only need to find **one** value that does not work
- Look out for the set of numbers for which the statement is made, it will often be just integers or natural numbers
- Numbers that have unusual results are often involved
  - It is often a good idea to try the values 0 and 1 first as they often behave in different ways to other numbers
  - The number 2 also behaves differently to other even numbers
    - It is the only even prime number
    - It is the only number that satisfies $n + n = n^n$
  - If it is the set of real numbers consider how rational and irrational numbers behave differently
  - Think about how positive and negative numbers behave differently
    - Particularly when working with inequalities



Exam Tip

- Read the question carefully, looking out for the set of numbers for which you need to prove the result

Exam Papers Practice

Worked example

For each of the following statements, show that they are false by giving a counterexample:

a) Given $n \in \mathbb{Z}^+$, if n^2 is a multiple of 4, then n is also a multiple of 4.

$n \in \mathbb{Z}^+ \leftarrow$ Set of positive integers only

We are only interested in positive integers so start by trying 1, 2 etc.

$1^2 = 1$ (not a multiple of 4)

$2^2 = 4$ $n^2 = 4$ (multiple of 4)

$n = 2$ (not a multiple of 4)

Let $n = 2$: $n^2 = 4$ (multiple of 4)

$n = 2$ (not a multiple of 4)

b) Given $x \in \mathbb{Z}$ then $3x$ is always greater than $2x$.

$x \in \mathbb{Z} \leftarrow$ Set of integers only

We are interested in both positive and negative integers and zero so consider how each of these groups behave:

Positive integers, e.g. let $x = 1$:

$2x = 2$

$3x = 3$ $\therefore 3x > 2x$

Zero:

Let $x = 0$

$2x = 0$

$3x = 0$ $\therefore$ $3x = 2x$ (this is enough to disprove the result)

Negative integers, e.g. let $x = -1$:

$2x = -2$

$3x = -3$ $\therefore 2x > 3x$ (Any negative integer can disprove the result)

Let $x = 0$

$2x = 0$

$3x = 0$ $\therefore$ $3x \not> 2x$