



1.3 Sequences & Series

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1.3.1 Language of Sequences & Series

Language of Sequences & Series

What is a sequence?

- A sequence is an ordered set of numbers with a rule for finding all of the numbers in the sequence
 - For example 1, 3, 5, 7, 9, ... is a sequence with the rule 'start at one and add two to each number'
- The numbers in a sequence are often called **terms**
- The terms of a sequence are often referred to by letters with a subscript
 - In IB this will be the letter u
 - So in the sequence above, $u_1 = 1$, $u_2 = 3$, $u_3 = 5$ and so on
- Each term in a sequence can be found by **substituting** the term number into **formula for the nth term**

What is a series?

- You get a series by summing up the terms in a sequence
 - E.g. For the sequence 1, 3, 5, 7, ... the associated series is 1 + 3 + 5 + 7 + ...
- We use the notation S_n to refer to the sum of the first *n* terms in the series
 - $S_n = u_1 + u_2 + u_3 + \dots + u_n$
 - So for the series above S₅ = 1 + 3 + 5 + 7 + 9 = 25



Determine the first five terms and the value of S_5 in the sequence with terms defined by $u_n = 5 - 2n$.

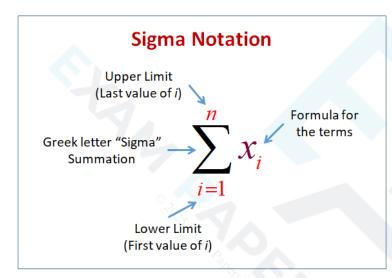
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U_{n} = 5 - 2n
find the term you
want by replacing
n with it's value.
first \Rightarrow U_{1} = 5 - 2(1) = 3 recognise the pattern.
U_{2} = 5 - 2(2) = 1
U_{3} = 5 - 2(3) = -12^{-2}
U_{3} = 5 - 2(3) = -12^{-2}
U_{4} = 5 - 2(4) = -32^{-2} = -72^{-2} subtract 2
U_{5} = 5 - 2(5) = -5
'start with 3 and subtract 2 from each number'.
S_{5} = 3 + 1 + (-1) + (-3) + (-5) = -5
the sum of
the first 5 terms
3, 1, -1, -3, -5
S_{5} = -5
```



Sigma Notation

What is sigma notation?

- Sigma notation is used to show the sum of a certain number of terms in a sequence
- The symbol Σ is the capital Greek letter sigma
- Σ stands for 'sum'
 - The expression to the right of the Σ tells you what is being summed, and the limits above and below tell you which terms you are summing



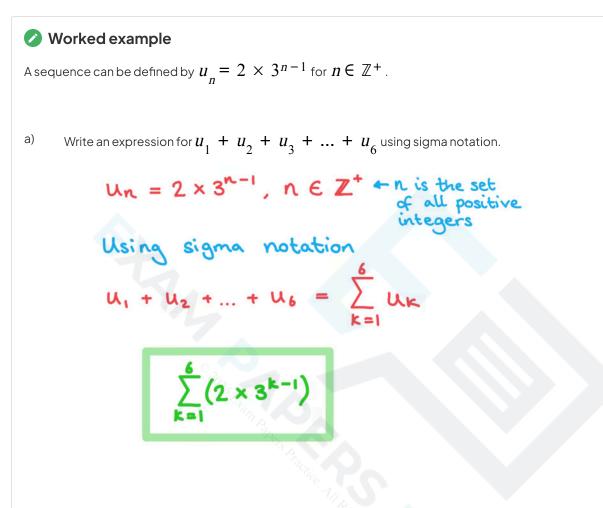
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Be careful, the limits don't have to start with 1

• For example
$$\sum_{k=0}^{4} (2k+1)$$
 or $\sum_{k=7}^{14} (2k-13)$

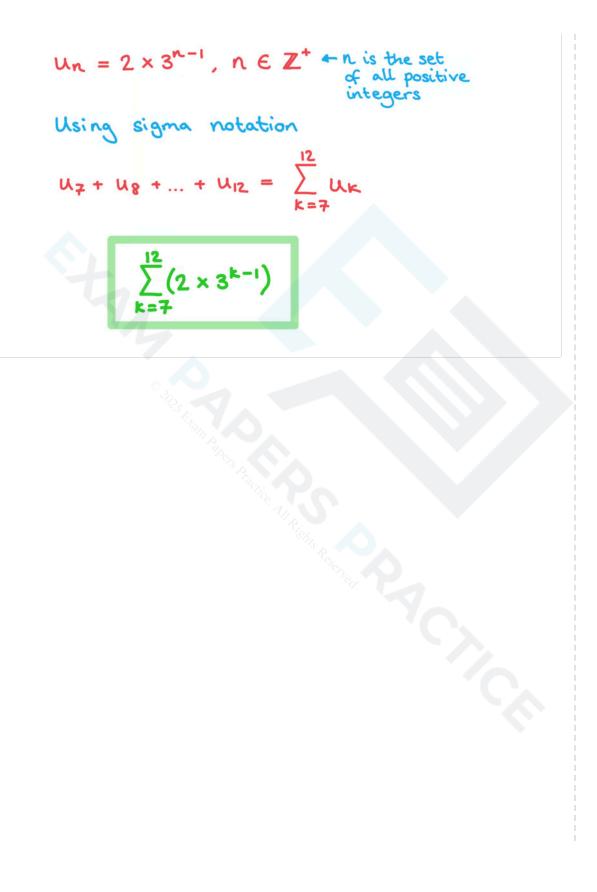
• r and k are commonly used variables within sigma notation





b) Write an expression for $u_7 + u_8 + u_9 + \dots + u_{12}$ using sigma notation.







1.3.2 Arithmetic Sequences & Series

Arithmetic Sequences

What is an arithmetic sequence?

- In an arithmetic sequence, the difference between consecutive terms in the sequence is constant
- This constant difference is known as the common difference, d, of the sequence
 - For example, 1, 4, 7, 10, ... is an arithmetic sequence with the rule 'start at one and add three to each number'
 - The first term, u₁, is 1
 - The common difference, d, is 3
 - An arithmetic sequence can be increasing (positive common difference) or decreasing (negative common difference)
 - Each term of an arithmetic sequence is referred to by the letter *u* with a subscript determining its place in the sequence

How do I find a term in an arithmetic sequence?

• The nth term formula for an arithmetic sequence is given as

$$u_n = u_1 + (n-1)d$$

- Where u_1 is the first term, and d is the common difference
- This is given in the formula booklet, you do not need to know how to derive it
- Enter the information you have into the formula and use your GDC to find the value of the term
- Sometimes you will be given a term and asked to find the first term or the common difference
- Substitute the information into the formula and solve the equation
 - You could use your GDC for this
- Sometimes you will be given two terms and asked to find both the first term and the common difference
 - Substitute the information into the formula and set up a system of linear equations
 - Solve the simultaneous equations
 - You could use your GDC for this





The fourth term of an arithmetic sequence is 10 and the ninth term is 25, find the first term and the common difference of the sequence.

> $U_4 = 10$, $U_9 = 25$ Formula for nth term of an arithmetic series: $U_n = U_1 + (n - 1)d$ Sub in $U_4 = 10$ and $U_9 = 25$ $U_4 = U_1 + (4 - 1)d = U_1 + 3d = 10$ $U_q = U_1 + (q - 1)d = U_1 + 8d = 25$ Solve using GOC : let u, = x and d = y $\alpha + 3y = 10$ x + 8y = 25x = 1, y = 3



Arithmetic Series

How do I find the sum of an arithmetic series?

- An arithmetic series is the sum of the terms in an arithmetic sequence
 - For the arithmetic sequence 1, 4, 7, 10, ... the arithmetic series is 1 + 4 + 7 + 10 + ...
- Use the following formulae to find the sum of the first *n* terms of the arithmetic series:

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$
; $S_n = \frac{n}{2}(u_1 + u_n)$

- u_1 is the first term
- *d* is the common difference
- *U_n* is the last term
- Both formulae are given in the formula booklet, you do not need to know how to derive them
- You can use whichever formula is more convenient for a given question
 - If you know the first term and common difference use the first version
 - If you know the first and last term then the second version is easier to use
- A question will often give you the sum of a certain number of terms and ask you to find the value of the first term or the common difference
 - Substitute the information into the formula and solve the equation
 - You could use your GDC for this



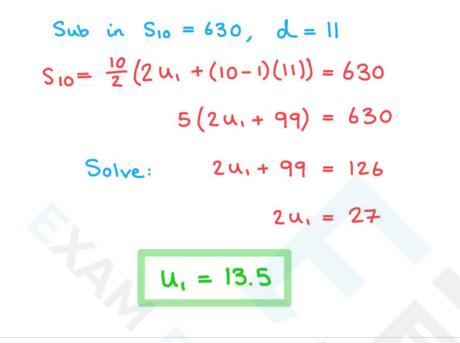
The sum of the first 10 terms of an arithmetic sequence is 630.

Find the common difference, d, of the sequence if the first term is 18. a)

> $S_{10} = 630$ Formula for the sum of an arithmetic series: $Sn = \frac{n}{2}(2u_1 + (n-1)d)$ Sub in $S_{10} = 630$, $U_1 = 18$ $S_{10} = \frac{10}{2} (2(18) + (10 - 1)d) = 630$ 5(36+9d) = 630Solve: 36 + 9d = 126 9d = 90 d = 10 No. d = 10

b) Find the first term of the sequence if the common difference, d, is 11.







1.3.3 Geometric Sequences & Series

Geometric Sequences

What is a geometric sequence?

- In a **geometric sequence**, there is a **common ratio**, **r**, between consecutive terms in the sequence
 - For example, 2, 6, 18, 54, 162, ... is a sequence with the rule 'start at two and multiply each number by three'
 - The first term, u₁, is 2
 - The common ratio, r, is 3
- A geometric sequence can be **increasing** (r > 1) or **decreasing** (0 < r < 1)
- If the common ratio is a **negative number** the terms will alternate between positive and negative values
 - For example, 1, -4, 16, -64, 256, ... is a sequence with the rule 'start at one and multiply each number by negative four'
 - The first term, u₁, is 1
 - The common ratio, r, is -4
- Each term of a geometric sequence is referred to by the letter *u* with a subscript determining its place in the sequence

How do I find a term in a geometric sequence?

• The n^{th} term formula for a geometric sequence is given as

$$u_n = u_1 r^{n-1}$$

- Where U_1 is the first term, and I is the common ratio
- This formula allows you to find **any term** in the geometric sequence
- It is given in the formula booklet, you do not need to know how to derive it
- Enter the information you have into the formula and use your GDC to find the value of the term
- Sometimes you will be given a term and asked to find the first term or the common ratio
 - Substitute the information into the formula and solve the equation
 You could use your CDC for this
 - You could use your GDC for this
- Sometimes you will be given two or more consecutive terms and asked to find both the first term and the common ratio
 - Find the common ratio by dividing a term by the one before it
 - Substitute this and one of the terms into the formula to find the first term
- Sometimes you may be given a term and the formula for the nth term and asked to find the value of n
 - You can solve these using **logarithms** on your GDC



The sixth term, $m{u}_6$, of a geometric sequence is 486 and the seventh term, $m{u}_7$, is 1458.

Find,

i) the common ratio, *I*, of the sequence,

$$u_{6} = 486$$
, $u_{7} = 1458$
The common ratio, r, is given by
 $r = \frac{u_{2}}{u_{1}} = \frac{u_{3}}{u_{2}} = \dots = \frac{u_{n+1}}{u_{n}}$
Sub in $u_{6} = 486$, $u_{7} = 1458$
 $r = \frac{u_{7}}{u_{6}} = \frac{1458}{486} = 3$
r = 3

ii) the first term of the sequence, \boldsymbol{u}_1 .



Formula for nth term of a geometric series: $U_{n} = U_{1} r^{n-1}$ Sub in r=3 and either U_{6} = 486 or U_{7} = 1458 $U_{6} = U_{1}(3)^{6-1} = 486$ Solve: 243 U₁ = 486 $U_{1} = 2$ $U_{1} = Z$



Geometric Series

How do I find the sum of a geometric series?

- A **geometric series** is the sum of a certain number of terms in a **geometric sequence**
 - For the geometric sequence 2, 6, 18, 54, ... the geometric series is 2 + 6 + 18 + 54 + ...
- The following formulae will let you find the sum of the first *n* terms of a geometric series:

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}$$

- u_1 is the first term
- I is the common ratio
- Both formulae are given in the formula booklet, you do not need to know how to derive them
- You can use whichever formula is more convenient for a given question
 - The first version of the formula is more convenient if r > 1 and the second is more convenient if r < 1
- A question will often give you the sum of a certain number of terms and ask you to find the value of the first term, the common ratio, or the number of terms within the sequence
 - Substitute the information into the formula and solve the equation
 - You could use your GDC for this



Worked example A geometric sequence has $u_1 = 25$ and r = 0.8. Find the value of u_5 and S_5 . $u_1 = 25$, r = 0.8Formula for not term of a geometric series: $U_n = U_1 r^{n-1}$ Sub in $U_1 = 25$, r = 0.8 $u_5 = 25(0.8)^4 = 10.24$ Formula for the sum of a geometric series: $S_n = \frac{U_1(r^n - 1)}{r - 1} = \frac{U_1(1 - r^n)}{1 - r} \eta$ Sub in $u_1 = 25$, r = 0.8 easier to use. $S_5 = \frac{U_1(1-r^5)}{1-r} = \frac{25(1-0.8^5)}{1-0.8} = 84.04$ $U_5 = 10.24$ $S_5 = 84.04$



Sum to Infinity

What is the sum to infinity of a geometric series?

- A geometric sequence will either increase or decrease away from zero or the terms will get progressively closer to zero
 - Terms will get closer to zero if the common ratio, r, is between 1 and -1
- If the terms are getting closer to zero then the series is said to **converge**
 - This means that the sum of the series will approach a limiting value
 - As the number of terms increase, the sum of the terms will get closer to the limiting value

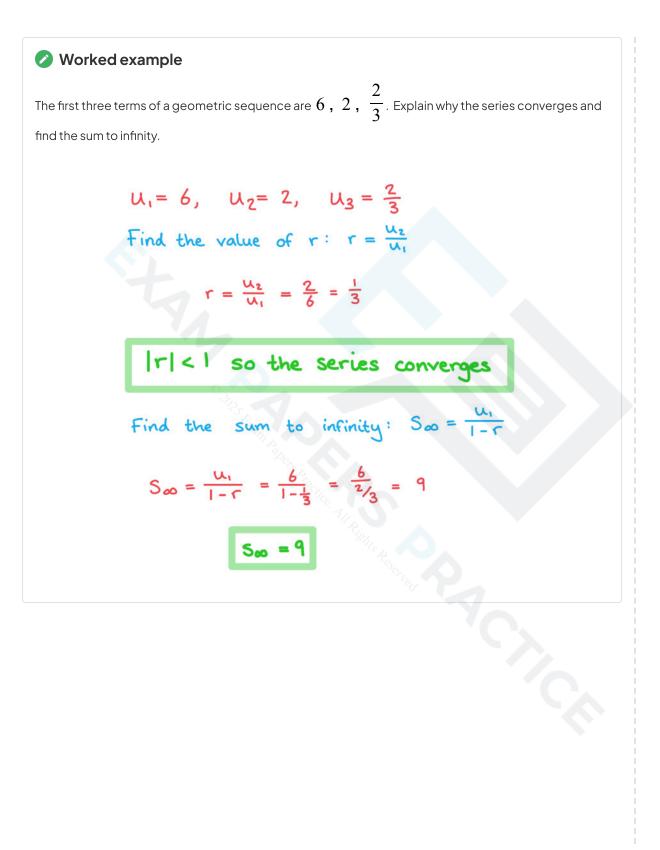
How do we calculate the sum to infinity?

- If asked to find out if a geometric sequence converges find the value of r
 - If |r| < 1 then the sequence converges
 - If $|r| \ge 1$ then the sequence does not converge and the sum to infinity cannot be calculated
 - |r| < 1 means -1 < r < 1
- If |r| < 1, then the geometric series **converges** to a finite value given by the formula

$$S_{\infty} = \frac{u_1}{1-r}, \ |r| < 1$$

- u_1 is the first term
- *I* is the common ratio
- This is in the formula book, you do not need to remember it







1.3.4 Applications of Sequences & Series

Applications of Arithmetic Sequences & Series

Many real-life situations can be modelled using sequences and series, including but not limited to: patterns made when tiling floors; seating people around a table; the rate of change of a population; the spread of a virus and many more.

What do I need to know about applications of arithmetic sequences and series?

- If a quantity is changing repeatedly by having a fixed amount added to or subtracted from it then the use of arithmetic sequences and arithmetic series is appropriate to model the situation
 - If a sequence seems to fit the pattern of an arithmetic sequence it can be said to be **modelled** by an arithmetic sequence
 - The scenario can be **modelled** using the given information and the formulae from the formula booklet
- A common application of arithmetic sequences and series is **simple interest**
 - Simple interest is when an initial investment is made and then a percentage of the initial investment is added to this amount on a regular basis (usually per year)
- Arithmetic sequences can be used to make estimations about how something will change in the future





Jasper is saving for a new car. He puts USD \$100 into his savings account and then each month he puts in USD \$10 more than the month before. Jasper needs USD \$1200 for the car. Assuming no interest is added, find,

i) the amount Jasper has saved after four months,

> Identify the arithmetic sequence : $u_1 = 100$, d = 10After 4 months Jasper will have saved: $U_1 + U_2 + U_3 + U_4 = S_4$ Formula for the sum of an arithmetic series: $Sn = \frac{n}{2}(2u_1 + (n-1)d)$ $S_4 = \frac{4}{2}(2u_1 + (4-1)d)$ Sub in $U_1 = 100$ and d = 10 $S_4 = \frac{4}{2}(2(100)+(4-1)(10))$ r Reserved = 2(200 + 30)= 2(230) $S_4 = 460

the month in which Jasper reaches his goal of USD \$1200. ii)



Sub Sn = 1200,
$$U_1 = 100$$
, $d = 10$ into formula:
 $1200 = \frac{n}{2} (2(100) + (n-1)(0))$

Solve using algebraic solver on GDC:

n = 8.67... or n = -27.67... ^Cdisregard as n cannot be negative.

: S8 < 1200

Sq > 1200 reaches total in 9th month

Jasper will reach USD\$1200 in the 9th month.



Applications of Geometric Sequences & Series

What do I need to know about applications of geometric sequences and series?

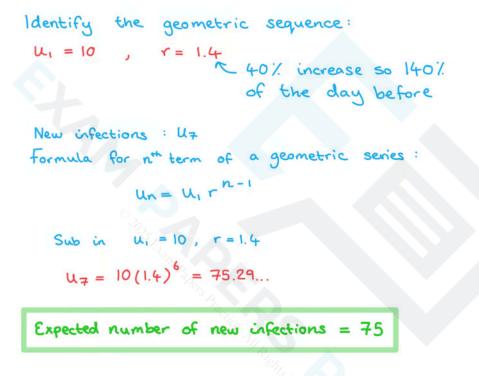
- If a quantity is changing repeatedly by a fixed percentage, or by being multiplied repeatedly by a fixed amount, then the use of geometric sequences and geometric series is appropriate to model the situation
 - If a sequence seems to fit the pattern of a geometric sequence it can be said to be **modelled** by a geometric sequence
 - The scenario can be modelled using the given information and the formulae from the formula booklet
- A common application of geometric sequences and series is **compound interest**
 - Compound interest is when an initial investment is made and then interest is paid on the initial amount **and on the interest already earned** on a regular basis (usually every year)
- Geometric sequences can be used to make estimations about how something will change in the future
- The questions won't always tell you to use sequences and series methods, so be prepared to spot 'hidden' sequences and series questions
 - Look out for questions on savings accounts, salaries, sales commissions, profits, population growth and decay, spread of bacteria etc





A new virus is circulating on a remote island. On day one there were 10 people infected, with the number of new infections increasing at a rate of 40% per day.

Find the expected number of people newly infected on the 7th day. a)



Find the expected number of infected people after one week (7 days), assuming no one has b) recovered yet.



Total infections: S_7 Formula for the sum of a geometric series: $S_n = \frac{U_1(r^n - 1)}{r - 1} \approx r > 1$ so this version is easier to use. Sub in $U_1 = 10$, r = 1.4 $S_7 = \frac{10(1.4^7 - 1)}{1.4 - 1} = 238.53...$

Expected number of total infections = 239

For more help, please visit www.exampaperspractice.co.uk



1.3.5 Compound Interest & Depreciation

Compound Interest

What is compound interest?

- Interest is a small percentage paid by a bank or company that is added on to an initial investment
 - Interest can also refer to an amount paid on a loan or debt, however IB compound interest questions will always refer to interest on investments
- Compound interest is where interest is paid on both the initial investment and any interest that has already been paid
 - Make sure you know the difference between compound interest and simple interest
 - Simple interest pays interest only on the initial investment
- The interest paid each time will increase as it is a percentage of a higher number
- Compound interest will be paid in instalments in a given timeframe
 - The interest rate, r, will be per annum (per year)
 - This could be written r % p.a.
 - Look out for phrases such as compounding annually (interest paid yearly) or compounding **monthly** (interest paid monthly)
 - If α^{0} % p.a. (per annum) is paid compounding monthly, then $\frac{\alpha}{12}$ % will be paid each month
 - The formula for compound interest allows for this so you do not have to compensate separately

How is compound interest calculated?

The formula for calculating compound interest is:

$$FV = PV \times \left(1 + \frac{r}{100k}\right)^k$$

- Where
 - FV is the future value
 - PV is the present value
 - *n* is the number of years
 - k is the number of compounding periods per year
 - r% is the nominal annual rate of interest
- N.C. This formula is given in the formula booklet, you do not have to remember it
- Be careful with the k value
 - Compounding annually means k = 1
 - Compounding half-yearly means k = 2
 - Compounding quarterly means k = 4
 - Compounding monthly means k = 12
- Your GDC will have a finance solver app on it which you can use to find the future value
 - This may also be called the TVM (time value of money) solver



- You will have to enter the information from the question into your calculator
- Be aware that many questions will be set up such that you will have to use the formula
 - So for compound interest questions it is better to use the formula from your formula booklet than your GDC



Kim invests MYR 2000 (Malaysian Ringgit) in an account that pays a nominal annual interest rate of 2.5% **compounded monthly**. Calculate the amount that Kim will have in her account after 5 years.

```
Compound interest formula:

FV = PV \left(1 + \frac{r}{100 k}\right)^{kn} \leftarrow number of years
future present Compounding

periods

Substitute values in:

PV = 2000 \text{ (initial investment)}
k = 12 \text{ (compounding monthly)}
r = 2.5%
n = 5 \text{ (number of years)}
FV = 2000 \left(1 + \frac{2.5}{(100)(12)}\right)^{(12\times5)}
= 2266.002...
FV \approx MYR 22270 \text{ (3sf)}
```



Depreciation

What is depreciation?

- Depreciation is when the value of something falls over time
- The most common examples of depreciation are the value of cars and technology
- If the depreciation is occurring at a constant rate then it is compound depreciation

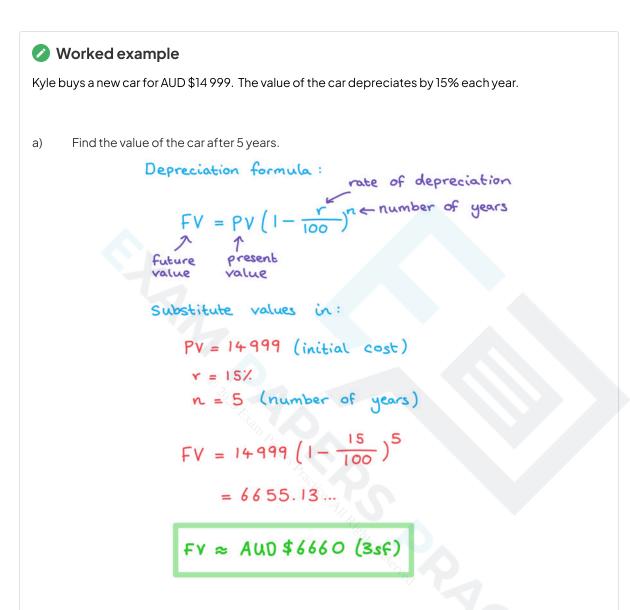
How is compound depreciation calculated?

• The formula for calculating compound depreciation is:

$$FV = PV \times \left(1 - \frac{r}{100}\right)^n$$

- Where
 - FV is the future value
 - PV is the present value
 - *n* is the number of years
 - *r%* is the rate of depreciation
- This formula is **not** given in the formula booklet, however it is almost the same as the formula for compound interest but
 - with a subtraction instead of an addition
 - the value of *k* will always be 1
- Your GDC **could** again be used to solve some compound depreciation questions, but watch out for those which are set up such that you will have to use the formula





b) Find the number of years and months it will take for the value of the car to be approximately AUD \$9999.



