



# 1.2 Sequences & Series

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### 1.2.1 Language of Sequences & Series

### Language of Sequences & Series

#### What is a sequence?

- A **sequence** is an ordered set of numbers with a well-defined rule for getting from one number to the next
  - For example 1, 3, 5, 7, 9, ... is a sequence with the rule 'start at one and add two to get each subsequent number'
- The numbers in a sequence are often called **terms**
- The terms of a sequence are often referred to by letters with a subscript
  - In IB this will be the letter u
  - So in the sequence above,  $u_1 = 1$ ,  $u_2 = 3$ ,  $u_3 = 5$  and so on
- Each term in a sequence can be found by substituting the term number into the formula for the n<sup>th</sup> term

#### What is a series?

- You get a **series** by summing up the terms in a sequence
  - E.g. For the sequence 1, 3, 5, 7, ... the associated series is 1+3+5+7+ ...
- We use the notation  $S_n$  to refer to the sum of the first n terms in the series
  - $S_n = u_1 + u_2 + u_3 + ... + u_n$
  - So for the series above  $S_5 = 1 + 3 + 5 + 7 + 9 = 25$





Determine the first five terms and the value of  $S_5$  in the sequence with terms defined by  $u_0 = 5 - 2n$ .

find the term you want by replacing n with it's value.

first term

$$U_1 = 5 - 2(1) = 3 \quad \text{recognise the pattern.}$$

$$U_2 = 5 - 2(2) = 1$$

$$U_3 = 5 - 2(3) = -1$$

$$U_4 = 5 - 2(4) = -3^2 - 2 \quad \text{rule is subtract 2}$$

'start with 3 and subtract 2 from each number'.

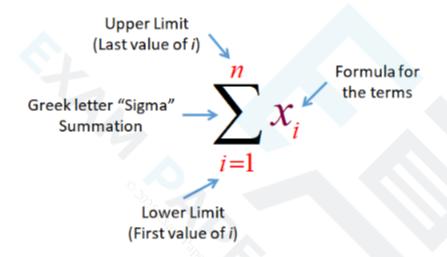
$$S_6 = 3 + 1 + (-1) + (-3) + (-5) = -5$$
e sum of
e first 5 terms



## **Sigma Notation**

#### What is sigma notation?

- Sigma notation is used to show the sum of a certain number of terms in a sequence
- The symbol ∑ is the capital Greek letter sigma
- Σ stands for 'sum'
  - The expression to the right of the  $\Sigma$  tells you what is being summed, and the limits above and below tell you which terms you are summing



■ Be careful, the limits don't have to start with 1

For example 
$$\sum_{k=0}^{4} (2k+1)$$
 or  $\sum_{k=7}^{14} (2k-13)$ 

ullet r and k are commonly used variables within sigma notation



A sequence can be defined by  $u_n = 2 \times 3^{n-1}$  for  $n \in \mathbb{Z}^+$  .

Write an expression for  $u_1 + u_2 + u_3 + \ldots + u_6$  using sigma notation. a)

Using sigma notation
$$u_1 + u_2 + ... + u_6 = \sum_{k=1}^{6} u_k$$

$$\sum_{k=1}^{6} (2 \times 3^{k-1})$$

Write an expression for  $u_7 + u_8 + u_9 + \dots + u_{12}$  using sigma notation. b)



Using sigma notation
$$u_7 + u_8 + ... + u_{12} = \sum_{k=7}^{12} u_k$$

$$\sum_{k=7}^{12} (2 \times 3^{k-1})$$



# 1.2.2 Arithmetic Sequences & Series

### **Arithmetic Sequences**

#### What is an arithmetic sequence?

- In an arithmetic sequence, the difference between consecutive terms in the sequence is constant
- This **constant difference** is known as the **common difference**, **d**, of the sequence
  - For example, 1, 4, 7, 10, ... is an arithmetic sequence with the rule 'start at one and add three to each number'
    - The first term,  $u_1$ , is 1
    - The **common difference**, **d**, is 3
  - An arithmetic sequence can be increasing (positive common difference) or decreasing (negative common difference)
  - Each term of an arithmetic sequence is referred to by the letter *u* with a subscript determining its place in the sequence

#### How do I find a term in an arithmetic sequence?

• The  $n^{th}$  term formula for an arithmetic sequence is given as

$$u_n = u_1 + (n-1)d$$

- ullet Where  $u_1$  is the first term, and d is the common difference
- This is given in the formula booklet, you do not need to know how to derive it
- Enter the information you have into the formula and use your GDC to find the value of the term
- Sometimes you will be given a term and asked to find the first term or the common difference
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this
- Sometimes you will be given two terms and asked to find both the first term and the common difference
  - Substitute the information into the formula and set up a **system of linear equations**
  - Solve the simultaneous equations
    - You could use your GDC for this



The fourth term of an arithmetic sequence is 10 and the ninth term is 25, find the first term and the common difference of the sequence.

Formula for nth term of an arithmetic series:

$$U_1 = U_1 + (n-1)d$$

Sub in  $U_4 = 10$  and  $U_9 = 25$ 
 $U_4 = U_1 + (4-1)d = U_1 + 3d = 10$ 
 $U_9 = U_1 + (9-1)d = U_1 + 8d = 25$ 

Solve using  $COC$ :

 $COC$ :

$$u_1 = 1$$
 $d = 3$ 



### **Arithmetic Series**

#### How do I find the sum of an arithmetic series?

- An arithmetic series is the sum of the terms in an arithmetic sequence
  - For the arithmetic sequence 1, 4, 7, 10, ... the arithmetic series is 1 + 4 + 7 + 10 + ...
- Use the following formulae to find the sum of the first *n* terms of the arithmetic series:

$$S_n = \frac{n}{2}(2u_1 + (n-1)d)$$
 ;  $S_n = \frac{n}{2}(u_1 + u_n)$ 

- $u_1$  is the first term
- d is the common difference
- $u_n$  is the last term
- Both formulae are given in the formula booklet, you do not need to know how to derive them
- You can use whichever formula is more convenient for a given question
  - If you know the first term and common difference use the first version
  - If you know the first and last term then the second version is easier to use
- A question will often give you the sum of a certain number of terms and ask you to find the value of the first term or the common difference
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this



The sum of the first 10 terms of an arithmetic sequence is 630.

a) Find the common difference, d, of the sequence if the first term is 18.

Sin = 630

Formula for the sum of an arithmetic series:

$$Sn = \frac{\pi}{2}(2u_1 + (n-1)d)$$

Sub in Sin = 630,  $u_1 = 18$ 
 $Sin = \frac{10}{2}(2(18) + (10-1)d) = 630$ 
 $S(36 + 9d) = 630$ 

Solve:  $36 + 9d = 126$ 
 $9d = 90$ 
 $d = 10$ 
 $d = 10$ 

b) Find the first term of the sequence if the common difference, d, is 11.



Sub in 
$$S_{10} = 630$$
,  $d = 11$ 

$$S_{10} = \frac{10}{2}(2u_1 + (10-1)(11)) = 630$$

$$5(2u_1 + 99) = 630$$

$$Solve: 2u_1 + 99 = 126$$

$$2u_1 = 27$$

$$u_1 = 13.5$$



### 1.2.3 Geometric Sequences & Series

### **Geometric Sequences**

### What is a geometric sequence?

- In a **geometric sequence**, there is a **common ratio**, **r**, between consecutive terms in the sequence
  - For example, 2, 6, 18, 54, 162, ... is a sequence with the rule 'start at two and multiply each number by three'
    - The first term,  $u_1$ , is 2
    - The **common ratio**, *r*, is 3
- A geometric sequence can be increasing (r > 1) or decreasing (0 < r < 1)
- If the common ratio is a **negative number** the terms will alternate between positive and negative values
  - For example, 1, -4, 16, -64, 256, ... is a sequence with the rule 'start at one and multiply each number by negative four'
    - The first term,  $u_1$ , is 1
    - The **common ratio**, *r*, is -4
- Each term of a geometric sequence is referred to by the letter *u* with a subscript determining its place in the sequence

#### How do I find a term in a geometric sequence?

• The  $n^{th}$  term formula for a geometric sequence is given as

$$u_{n} = u_{1} r^{n-1}$$

- Where  $\boldsymbol{u}_1$  is the first term, and  $\boldsymbol{r}$  is the common ratio
- This formula allows you to find **any term** in the geometric sequence
- It is given in the formula booklet, you do not need to know how to derive it
- Enter the information you have into the formula and use your GDC to find the value of the term
- Sometimes you will be given a term and asked to find the first term or the common ratio
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this
- Sometimes you will be given two or more consecutive terms and asked to find both the first term and the common ratio
  - Find the common ratio by dividing a term by the one before it
  - Substitute this and one of the terms into the formula to find the first term
- Sometimes you may be given a term and the formula for the  $n^{th}$  term and asked to find the value of n
  - You can solve these using logarithms on your GDC



The sixth term,  $\boldsymbol{u}_6$ , of a geometric sequence is 486 and the seventh term,  $\boldsymbol{u}_7$ , is 1458.

Find,

i) the common ratio,  $\mathbf{r}$ , of the sequence,

The common ratio, r, is given by
$$r = \frac{U_2}{U_1} = \frac{U_3}{U_2} = \dots = \frac{U_{n+1}}{U_n}$$
Sub in  $U_6 = 486$ ,  $U_7 = 1458$ 

$$r = \frac{U_7}{U_6} = \frac{1458}{486} = 3$$

ii) the first term of the sequence,  $\boldsymbol{u}_1$  .



$$u_6 = u_1(3)^{6-1} = 486$$

$$u_i = 2$$



#### **Geometric Series**

#### How do I find the sum of a geometric series?

- A geometric series is the sum of a certain number of terms in a geometric sequence
  - For the geometric sequence 2, 6, 18, 54, ... the geometric series is 2 + 6 + 18 + 54 + ...
- The following formulae will let you find the sum of the first *n* terms of a geometric series:

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}$$

- $u_1$  is the first term
- *I* is the common ratio
- Both formulae are given in the formula booklet, you do not need to know how to derive them
- You can use whichever formula is more convenient for a given question
  - The first version of the formula is more convenient if r>1 and the second is more convenient if r<1
- A question will often give you the sum of a certain number of terms and ask you to find the value of the first term, the common ratio, or the number of terms within the sequence
  - Substitute the information into the formula and solve the equation
    - You could use your GDC for this



A geometric sequence has  $u_1=25$  and r=0.8 . Find the value of  $u_5$  and  $S_5$  .

$$u_1 = 25$$
,  $r = 0.8$ 

Formula for nth term of a geometric series:

Sub in 
$$U_1 = 25$$
,  $r = 0.8$ 

$$U_5 = 25(0.8)^4 = 10.24$$

Formula for the sum of a geometric series:

$$S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}$$

$$r < 1 \text{ so this}$$

Sub in 
$$U_1 = 25$$
,  $r = 0.8$  easier to use.

$$S_5 = \frac{U_1(1-r^5)}{1-r} = \frac{25(1-0.8^5)}{1-0.8} = 84.04$$

$$U_5 = 10.24$$
  
 $S_5 = 84.04$ 

$$S_5 = 84.04$$



### 1.2.4 Applications of Sequences & Series

### **Applications of Arithmetic Sequences & Series**

Many real-life situations can be modelled using sequences and series, including but not limited to: patterns made when tiling floors; seating people around a table; the rate of change of a population; the spread of a virus and many more.

#### What do I need to know about applications of arithmetic sequences and series?

- If a quantity is changing repeatedly by having a fixed amount added to or subtracted from it then the
  use of arithmetic sequences and arithmetic series is appropriate to model the situation
  - If a sequence seems to fit the pattern of an arithmetic sequence it can be said to be **modelled** by an arithmetic sequence
  - The scenario can be modelled using the given information and the formulae from the formula booklet
- A common application of arithmetic sequences and series is **simple interest** 
  - Simple interest is when an initial investment is made and then a percentage of the initial investment is added to this amount on a regular basis (usually per year)
- Arithmetic sequences can be used to make estimations about how something will change in the future



Jasper is saving for a new car. He puts USD \$100 into his savings account and then each month he puts in USD \$10 more than the month before. Jasper needs USD \$1200 for the car. Assuming no interest is added, find,

i) the amount Jasper has saved after four months,

Identify the arithmetic sequence: 
$$U_1 = 100 \quad , \quad d = 10$$
After 4 months Jasper will have saved: 
$$U_1 + U_2 + U_3 + U_4 = S_4$$
Formula for the sum of an arithmetic series: 
$$S_1 = \frac{\alpha}{2} \left( 2U_1 + (n-1)d \right)$$

$$S_4 = \frac{4}{2} \left( 2U_1 + (4-1)d \right)$$
Sub in  $U_1 = 100$  and  $d = 10$ 

$$S_4 = \frac{4}{2} \left( 2(100) + (4-1)(10) \right)$$

$$= 2 \left( 200 + 30 \right)$$

$$= 2 \left( 230 \right)$$

$$S_4 = $460$$
Then the properties of the properties of

the month in which Jasper reaches his goal of USD \$1200. ii)



Sub 
$$S_n = 1200$$
,  $U_1 = 100$ ,  $d = 10$  into formula:  
 $1200 = \frac{n}{2}(2(100) + (n-1)(0))$ 

Solve using algebraic solver on GDC:

n = 8.67... or n = -27.67... disregard as n cannot be negative.

:. Sg < 1200

Sg > 1200 reaches total in 9th month

Jasper will reach USD\$1200 in the 9th month.



### **Applications of Geometric Sequences & Series**

### What do I need to know about applications of geometric sequences and series?

- If a quantity is changing repeatedly by a fixed percentage, or by being multiplied repeatedly by a fixed amount, then the use of geometric sequences and geometric series is appropriate to model the situation
  - If a sequence seems to fit the pattern of a geometric sequence it can be said to be modelled by a geometric sequence
  - The scenario can be modelled using the given information and the formulae from the formula booklet
- A common application of geometric sequences and series is compound interest
  - Compound interest is when an initial investment is made and then interest is paid on the initial
    amount and on the interest already earned on a regular basis (usually every year)
- Geometric sequences can be used to make estimations about how something will change in the future
- The questions won't always tell you to use sequences and series methods, so be prepared to spot 'hidden' sequences and series questions
  - Look out for questions on savings accounts, salaries, sales commissions, profits, population growth and decay, spread of bacteria etc



A new virus is circulating on a remote island. On day one there were 10 people infected, with the number of new infections increasing at a rate of 40% per day.

a) Find the expected number of people newly infected on the 7<sup>th</sup> day.

Identify the geometric sequence:

$$u_1 = 10$$
,  $r = 1.4$ 
 $40\%$  increase so  $140\%$ 

of the day before

New infections:  $u_7$ 

Formula for  $n^{th}$  term of a geometric series:

 $u_1 = u_1 r^{1}$ 

Sub in  $u_1 = 10$ ,  $r = 1.4$ 
 $u_7 = 10(1.4)^6 = 75.29...$ 

Expected number of new infections = 75

b) Find the expected number of infected people after one week (7 days), assuming no one has recovered yet.



Total infections: S7

Formula for the sum of a geometric series:

$$S_n = \frac{U_1(r^n-1)}{r-1} \approx r > 1$$
 so this version is

Sub in 
$$u_1 = 10$$
,  $r = 1.4$ 

$$S_7 = \frac{10(1.4^7 - 1)}{1.4 - 1} = 238.53...$$

Expected number of total infections = 239