



1.2 Exponentials & Logs

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1.2.1 Introduction to Logarithms

Introduction to Logarithms

What are logarithms?

- A logarithm is the inverse of an exponent
 - If $a^x = b$ then $\log_a(b) = x$ where $a > 0, b > 0, a \ne 1$
 - This is in the formula booklet
 - The number a is called the **base** of the logarithm
 - Your GDC will be able to use this function to solve equations involving exponents
- Try to get used to 'reading' logarithm statements to yourself
 - $\log_a(b) = x$ would be read as "the power that you raise a to, to get b, is x"
 - So $\log_5 125 = 3$ would be read as "the power that you raise 5 to, to get 125, is 3"
- Two important cases are:
 - - Where e is the mathematical constant 2.718...
 - This is called the **natural logarithm** and will have its own button on your GDC
 - $\bullet \log x = \log_{10}(x)$
 - Logarithms of **base 10** are used often and so abbreviated to **log** x

Why use logarithms?

- Logarithms allow us to solve equations where the exponent is the unknown value
 - We can solve some of these by inspection
 - For example, for the equation $2^x = 8$ we know that x must be 3
 - Logarithms allow use to solve more complicated problems
 - For example, the equation $2^x = 10$ does not have a clear answer
 - $\,\blacksquare\,$ Instead, we can use our GDCs to find the value of $log_2 10$

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Solve the following equations:

i)
$$x = \log_3 27$$
,

$$\alpha = \log_3 27 \iff 3^{\infty} = 27$$

We can see from inspection:

$$3^3 = 27 \iff \infty = 3$$

OR: Use GDC to find answer directly.

ii)
$$2^x = 21.4$$
, giving your answer to 3 s.f.



$$2^{\infty} = 21.4$$
 This cannot be seen from inspection:

$$2^{\infty} = 21.4 \iff \infty = \log_2 21.4$$

use GDC to find answer directly.

$$\infty = 4.42 (3 s.f.)$$

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1.2.2 Laws of Logarithms

Laws of Logarithms

What are the laws of logarithms?

- Laws of logarithms allow you to simplify and manipulate expressions involving logarithms
 - The laws of logarithms are equivalent to the laws of indices
- The laws you need to know are, given a, x, y > 0:

$$\log_a xy = \log_a x + \log_a y$$

• This relates to $a^x \times a^y = a^{x+y}$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

• This relates to $a^x \div a^y = a^{x-y}$

$$\log_a x^m = m \log_a x$$

This relates to $(a^x)^y = a^{xy}$

Useful results from the laws of logarithms

• Given
$$a > 0$$
, $a \neq 1$

$$\log_a 1 = 0$$

- This is equivalent to $a^0 = 1$
- If we substitute b for a into the given identity in the formula booklet

•
$$a^x = b \Leftrightarrow \log_a b = x \text{ where } a > 0, b > 0, a \neq 1$$

•
$$a^x = a \Leftrightarrow \log_a a = x \text{ gives } a^1 = a \Leftrightarrow \log_a a = 1$$

- This is an important and useful result
- Substituting this into the third law gives the result

$$\log_a a^k = k$$

Taking the inverse of its operation gives the result

$$a^{\log_a x} = x$$

• From the third law we can also conclude that

$$\log_a \frac{1}{x} = -\log_a x$$



- These useful results are **not** in **the formula booklet** but can be deduced from the laws that are
- Beware...

$$\log_a(x+y) \neq \log_a x + \log_a y$$

- lacktriangledown These results apply to $\ln x \left(\log_e x\right)$ too
 - Two particularly useful results are

$$e^{\ln x} = x$$

- Laws of logarithms can be used to ...
 - simplify expressions
 - solve logarithmic equations
 - solve exponential equations

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a) Write the expression $2 \log 4 - \log 2$ in the form $\log k$, where $k \in \mathbb{Z}$.

Using the law
$$\log_a x^m = m\log_a x$$

 $2\log 4 = \log 4^2 = \log 16$
 $2\log 4 - \log 2 = \log 4^2 - \log 2$
 $= \log 16 - \log 2$
Using the law $\log_a \frac{x}{y} = \log_a x - \log_a y$
 $\log 16 - \log 2 = \log \frac{16}{2} = \log 8$

b) Hence, or otherwise, solve $2 \log 4 - \log 2 = -\log \frac{1}{x}$.

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To solve
$$2\log 4 - \log 2 = \log \frac{1}{x}$$
 rewrite as
$$\log 8 = -\log \frac{1}{x}$$
from part (a)

Use the index law $\frac{1}{x} = x^{-1}$

$$\log 8 = -\log x^{-1}$$

$$\log 8 = \log x$$

$$8 = x$$

$$x = 8$$

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Change of Base

Why change the base of a logarithm?

- The laws of logarithms can only be used if the logs have the same **base**
 - If a problem involves logarithms with different bases, you can change the base of the logarithm and then apply the laws of logarithms
- Changing the base of a logarithm can be particularly useful if you need to evaluate a log problem without a calculator
 - Choose the base such that you would know how to solve the problem from the equivalent exponent

How do I change the base of a logarithm?

• The formula for changing the base of a logarithm is

$$\log_a x = \frac{\log_b x}{\log_b a}$$

- This is in the formula booklet
- The value you choose for b does not matter, however if you do not have a calculator, you can choose b such that the problem will be possible to solve

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By choosing a suitable value for b, use the change of base law to find the value of $\log_8 32$ without using a calculator.

Change of base law:
$$\log_{a} x = \frac{\log_{b} x}{\log_{b} a}$$

$$\log_{a} 32$$

$$2^{3} = 8$$

Chaose b = 2 to allow for a solution by inspection

$$\log_8 32 = \frac{\log_2 32}{\log_2 8} = \frac{5}{3}$$



1.2.3 Solving Exponential Equations

Solving Exponential Equations

What are exponential equations?

- An exponential equation is an equation where the unknown is a power
 - In simple cases the solution can be spotted without the use of a calculator
 - For example,

$$5^{2x} = 125$$

$$2x = 3$$

$$x = \frac{3}{2}$$

- In more complicated cases the laws of logarithms should be used to solve exponential equations
- The **change of base** law can be used to solve some exponential equations without a calculator
 - For example,

$$27^{x} = 9$$

$$x = \log_{27} 9$$

$$= \frac{\log_{3} 9}{\log_{3} 27}$$

$$= \frac{2}{\log_{3} 27}$$

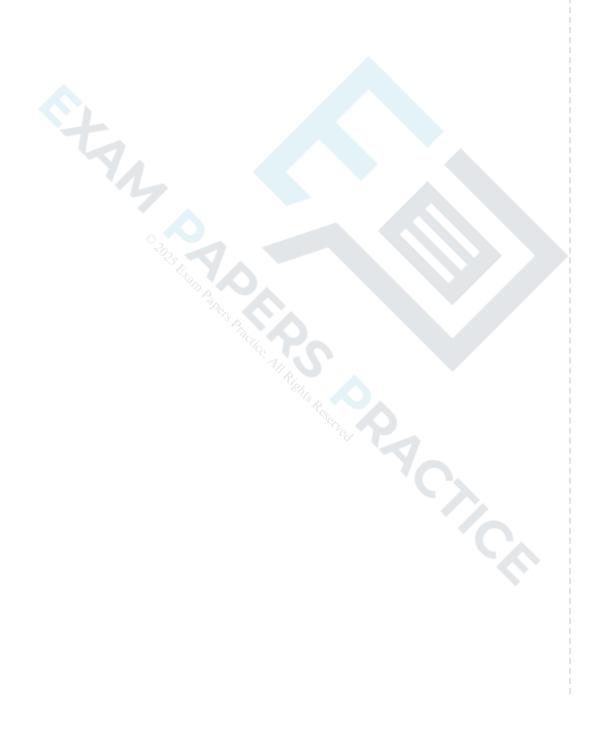
How do we use logarithms to solve exponential equations?

- An exponential equation can be solved by taking logarithms of both sides
- The **laws of indices** may be needed to rewrite the equation first
- The laws of logarithms can then be used to solve the equation
 - In (log_e) is often used
 - The answer is often written in terms of In
- A question my ask you to give your answer in a particular form
- Follow these steps to solve exponential equations
 - STEP 1: Take logarithms of both sides
 - STEP 2: Use the laws of logarithms to remove the powers
 - STEP 3: Rearrange to isolate *x*
 - STEP 4: Use logarithms to solve for x

What about hidden quadratics?



- Look for hidden squared terms that could be changed to form a quadratic
 - In particular look out for terms such as
 - $4^x = (2^2)^x = 2^{2x} = (2^x)^2$
 - $e^{2x} = (e^2)^x = (e^x)^2$





Solve the equation $4^x - 3(2^{x+1}) + 9 = 0$. Give your answer correct to three significant figures.

Spot the hidden quadratic:
$$4^{\infty} = (2^2)^{\infty} = (2^{\infty})^2$$

By the laws of indices $2^{\infty+1} = 2^{\infty} \times 2^1$
 $(2^{\infty})^2 - 3(2^{\infty+1}) + 9 = 0$
 $= 2 \times 2^{\infty}$
 $(2^{\infty})^2 - 3 \times 2 \times 2^{\infty} + 9 = 0$
 $(2^{\infty})^2 - 6 \times 2^{\infty} + 9 = 0$

Let $u = 2^{\infty}$ $u^2 - 6u + 9 = 0$
 $(u - 3)(u - 3) = 0$
 $u = 3$ $\therefore 2^{\infty} = 3$

Solve the exponential equation $2^{\infty} = 3$

Step 1: Take Logarithms of both sides: $\ln(2^{\infty}) = \ln(3)$

Step 2: Use the law $\log_a x^m = m\log_a x$ $x \ln 2 = \ln 3$

Step 3: Rearrange to isolate $x = \frac{\ln 3}{\ln 2}$

Step 4: Solve

 $x = \frac{\ln 3}{\ln 2} = 1.584...$