DP IB Maths: AI HL

1.2 Exponentials & Logs

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1.2.1 Exponents

Laws of Indices

What are the laws of indices?

- Laws of indices (or index laws) allow you to simplify and manipulate expressions involving exponents
  - An exponent is a power that a number (called the base) is raised to
  - Laws of indices can be used when the numbers are written with the same base
- The index laws you need to know are:
  - $(xy)^m = x^m y^m$
  - $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$
  - $x^m \times x^n = x^{m+n}$
  - $x^m \div x^n = x^{m-n}$
  - $(x^m)^n = x^{mn}$
  - $x^1 = x$
  - $x^0 = 1$
  - $\frac{1}{x^m} = x^{-m}$
  - $x^{\frac{1}{n}} = \sqrt[n]{x}$
  - $x^{\frac{m}{n}} = \sqrt[n]{x^m}$
- These laws are **not in the formula booklet** so you must remember them

How are laws of indices used?

- You will need to be able to carry out multiple calculations with the laws of indices
  - Take your time and apply each law individually
  - Work with numbers first and then with algebra
- Index laws only work with terms that have the same base, make sure you **change the base** of the term before using any of the index laws
  - Changing the base means rewriting the number as an exponent with the base you need
  - For example, $9^4 = (3^2)^4 = 3^{2 \times 4} = 3^8$
  - Using the above can them help with problems like $9^4 \div 3^7 = 3^8 \div 3^7 = 3^1 = 3$

Worked example

Simplify the following equations:

i) $\dfrac{(3x^2)(2x^3y^2)}{(6x^2y)}$.

Apply each law separately:

$3 \times 2 = 6$

$$\frac{(3x^2)(2x^3y^2)}{6x^2y}$$

expand numerator

$x^2 \times x^3 = x^5$

$$\frac{(6x^2)(x^3y^2)}{6x^2y}$$

cancelling

$$\frac{\cancel{6}x^5y^2}{\cancel{6}x^2y}$$

$x^5 \div x^2 = x^{5-2} = x^3$

$y^2 \div y = y^{2-1} = y$

$$x^3y$$

$$\boxed{\frac{(3x^2)(2x^3y^2)}{6x^2y} = x^3y}$$

ii) $(4x^2y^{-4})^3(2x^3y^{-1})^{-2}$.

$$(4x^2y^{-4})^3(2x^3y^{-1})^{-2}$$

Rewrite as a fraction

$$\frac{(4x^2y^{-4})^3}{(2x^3y^{-1})^2}$$

expand numerator and denominator

$$\frac{64x^6y^{-12}}{4x^6y^{-2}}$$

cancelling

$$\frac{\cancel{64}\,\cancel{x^6}\,y^2}{\cancel{4}\,\cancel{x^6}\,y^{12}}$$

The negative exponents can be rewritten as their reciprocals

$$16y^{-10}$$

$$\boxed{\frac{16}{y^{10}}}$$

1.2.2 Logarithms

Introduction to Logarithms

What are logarithms?

- A logarithm is the inverse of an exponent
  - If $a^x = b$ then $\log_a(b) = x$ where $a > 0$, $b > 0$, $a \neq 1$
    - This is in the formula booklet
    - The number a is called the **base** of the logarithm
    - Your GDC will be able to use this function to solve equations involving exponents
- Try to get used to 'reading' logarithm statements to yourself
  - $\log_a(b) = x$ would be read as "the power that you raise a to, to get b, is x"
  - So $\log_5 125 = 3$ would be read as "the power that you raise 5 to, to get 125, is 3"
- Two important cases are:
  - $\ln x = \log_e(x)$
    - Where e is the mathematical constant 2.718…
    - This is called the **natural logarithm** and will have its own button on your GDC
  - $\log x = \log_{10}(x)$
    - Logarithms of **base 10** are used often and so abbreviated to **log *x***

Why use logarithms?

- Logarithms allow us to solve equations where the exponent is the unknown value
  - We can solve some of these by inspection
    - For example, for the equation $2^x = 8$ we know that x must be 3
  - Logarithms allow use to solve more complicated problems
    - For example, the equation $2^x = 10$ does not have a clear answer
    - Instead, we can use our GDCs to find the value of $\log_2 10$

Worked example

Solve the following equations:

i) $x = \log_3 27$,

$x = \log_3 27 \iff 3^x = 27$

We can see from inspection:

$3^3 = 27 \iff x = 3$

$\boxed{x = 3}$

OR: Use GDC to find answer directly.

ii) $2^x = 21.4$, giving your answer to 3 s.f.

$2^x = 21.4$ This cannot be seen from inspection:

$$2^x = 21.4 \iff x = \log_2 21.4$$

Use GDC to find answer directly.

$$\log_2 21.4 = 4.4195\ldots$$

$$\boxed{x = 4.42 \text{ (3 s.f.)}}$$

Laws of Logarithms

What are the laws of logarithms?

- Laws of logarithms allow you to simplify and manipulate expressions involving logarithms
  - The laws of logarithms are equivalent to the **laws of indices**
- The laws you need to know are, given $a, x, y > 0$:
  - $\log_a xy = \log_a x + \log_a y$
    - This relates to $a^x \times a^y = a^{x+y}$
  - $\log_a \frac{x}{y} = \log_a x - \log_a y$
    - This relates to $a^x \div a^y = a^{x-y}$
  - $\log_a x^m = m\log_a x$
    - This relates to $(a^x)^y = a^{xy}$
- These laws are **in the formula booklet** so you do not need to remember them
  - You must make sure you know how to use them

Useful results from the laws of logarithms

- Given $a > 0, a \neq 1$
  - $\log_a 1 = 0$
    - This is equivalent to $a^0 = 1$
- If we substitute *b* for *a* into the given identity in the formula booklet
  - $a^x = b \Leftrightarrow \log_a b = x$ where $a > 0, b > 0, a \neq 1$
  - $a^x = a \Leftrightarrow \log_a a = x$ gives $a^1 = a \Leftrightarrow \log_a a = 1$
    - This is an important and useful result

  - Substituting this into the third law gives the result
    - $\log_a a^k = k$
  - Taking the inverse of its operation gives the result
    - $a^{\log_a x} = x$
  - From the third law we can also conclude that
    - $\log_a \frac{1}{x} = -\log_a x$

- These useful results are **not in the formula booklet** but can be deduced from the laws that are
- Beware…
  - …$\log_a(x+y) \neq \log_a x + \log_a y$
- These results apply to $\ln x$ $(\log_e x)$ too
  - Two particularly useful results are
    - $\ln e^x = x$
    - $e^{\ln x} = x$
- Laws of logarithms can be used to …
  - simplify expressions
  - solve logarithmic equations
  - solve exponential equations

Worked example

a)

Write the expression $2\log 4 - \log 2$ in the form $\log k$, where $k \in \mathbb{Z}$.

Using the law $\log_a x^m = m\log_a x$

$2\log 4 = \log 4^2 = \log 16$

$2\log 4 - \log 2 = \log 4^2 - \log 2$

$= \log 16 - \log 2$

Using the law $\log_a \frac{x}{y} = \log_a x - \log_a y$

$\log 16 - \log 2 = \log \frac{16}{2} = \log 8$

$$\boxed{2\log 4 - \log 2 = \log 8}$$

b) Hence, or otherwise, solve $2\log 4 - \log 2 = -\log \frac{1}{x}$.

To solve $2\log 4 - \log 2 = \log \frac{1}{x}$ rewrite as

$$\log 8 = -\log \frac{1}{x}$$

from part (a)

Use the index law $\frac{1}{x} = x^{-1}$

$$\log 8 = -\log x^{-1}$$

$$\log 8 = \log x$$

$\log_a x^m = m\log_a x$

$$8 = x$$

$$\boxed{x = 8}$$