



# 1.2 Exponentials & Logs

### Contents

- ✤ 1.2.1 Introduction to Logarithms
- ✤ 1.2.2 Laws of Logarithms
- \* 1.2.3 Solving Exponential Equations



## 1.2.1 Introduction to Logarithms

### Introduction to Logarithms

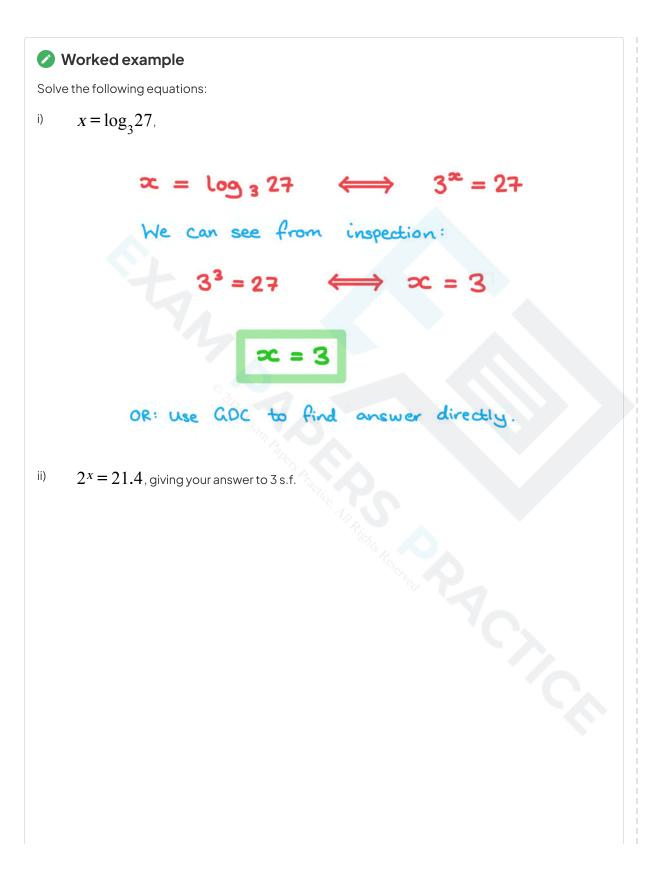
#### What are logarithms?

- A logarithm is the inverse of an exponent
  - If  $a^x = b$  then  $\log_a(b) = x$  where  $a > 0, b > 0, a \neq 1$ 
    - This is in the formula booklet
    - The number *a* is called the **base** of the logarithm
    - Your GDC will be able to use this function to solve equations involving exponents
- Try to get used to 'reading' logarithm statements to yourself
  - $\log_a(b) = x$  would be read as "the power that you raise a to, to get b, is x"
  - So  $\log_5 125 = 3$  would be read as "the power that you raise 5 to, to get 125, is 3"
- Two important cases are:
  - $\ln x = \log_{e}(x)$ 
    - Where e is the mathematical constant 2.718...
    - This is called the **natural logarithm** and will have its own button on your GDC
  - $\log x = \log_{10}(x)$ 
    - Logarithms of **base 10** are used often and so abbreviated to **log x**

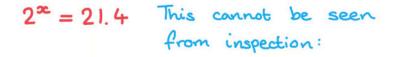
#### Why use logarithms?

- Logarithms allow us to solve equations where the exponent is the unknown value
  - We can solve some of these by inspection
    - For example, for the equation  $2^x = 8$  we know that x must be 3
  - Logarithms allow use to solve more complicated problems
    - For example, the equation  $2^x = 10$  does not have a clear answer
    - Instead, we can use our GDCs to find the value of  $\log_2 10$









# $2^{\infty} = 21.4 \iff \infty = \log_2 21.4$

use GDC to find answer directly.

 $log_2 21.4 = 4.4195...$ 

 $\infty = 4.42$  (3 s.f.)



## 1.2.2 Laws of Logarithms

### Laws of Logarithms

#### What are the laws of logarithms?

- Laws of logarithms allow you to simplify and manipulate expressions involving logarithms
  - The laws of logarithms are equivalent to the **laws of indices**
- The laws you need to know are, given a, x, y > 0:

$$\log_a xy = \log_a x + \log_a y$$

• This relates to  $a^x \times a^y = a^{x+y}$ 

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

• This relates to  $a^x \div a^y = a^{x-y}$ 

$$\log_a x^m = m \log_a x$$

- This relates to  $(a^x)^y = a^{xy}$
- These laws are in the formula booklet so you do not need to remember them
  - You must make sure you know how to use them

#### Useful results from the laws of logarithms

- Given a > 0,  $a \neq 1$ 
  - $\log_{a} 1 = 0$ 
    - This is equivalent to  $a^0 = 1$
- If we substitute b for a into the given identity in the formula booklet

$$a^x = b \iff \log_a b = x$$
 where  $a > 0, b > 0, a \neq 1$ 

• 
$$a^x = a \Leftrightarrow \log_a a = x$$
 gives  $a^1 = a \Leftrightarrow \log_a a = 1$ 

- This is an important and useful result
- Substituting this into the third law gives the result

$$\log_a a^k = k$$

Taking the inverse of its operation gives the result

$$a^{\log_a x} = x$$

From the third law we can also conclude that

$$\log_a \frac{1}{x} = -\log_a x$$



- These useful results are **not in the formula booklet** but can be deduced from the laws that are
- Beware...
  - $\log_a(x+y) \neq \log_a x + \log_a y$
- These results apply to  $\ln x (\log_e x)$  too
  - Two particularly useful results are
    - $\ln e^x = x$
    - $e^{\ln x} = x$
- Laws of logarithms can be used to ...
  - simplify expressions
  - solve logarithmic equations
  - solve exponential equations



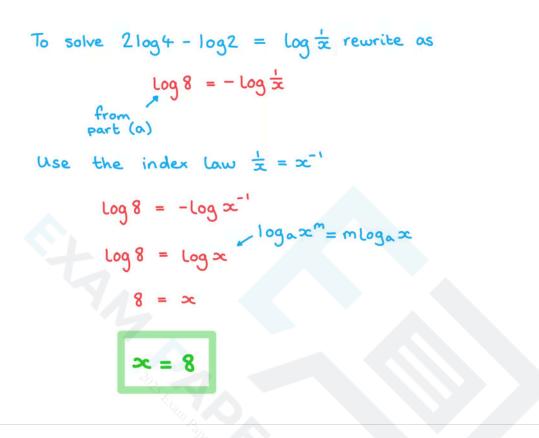


a) Write the expression  $2 \log 4 - \log 2$  in the form  $\log k$ , where  $k \in \mathbb{Z}$ .

Using the law 
$$\log_{a} x^{m} = m \log_{a} x$$
  
 $2\log 4 = \log 4^{2} = \log 16$   
 $2\log 4 - \log 2 = \log 4^{2} - \log 2$   
 $= \log 16 - \log 2$   
Using the law  $\log_{a} \frac{x}{y} = \log_{a} x - \log_{a} y$   
 $\log 16 - \log 2 = \log \frac{16}{2} = \log 8$   
 $2\log 4 - \log 2 = \log \frac{16}{2} = \log 8$ 

b) Hence, or otherwise, solve  $2 \log 4 - \log 2 = -\log \frac{1}{x}$ .







### Change of Base

#### Why change the base of a logarithm?

- The laws of logarithms can only be used if the logs have the same **base** 
  - If a problem involves logarithms with different bases, you can change the base of the logarithm and then apply the laws of logarithms
- Changing the base of a logarithm can be particularly useful if you need to evaluate a log problem without a calculator
  - Choose the base such that you would know how to solve the problem from the equivalent exponent

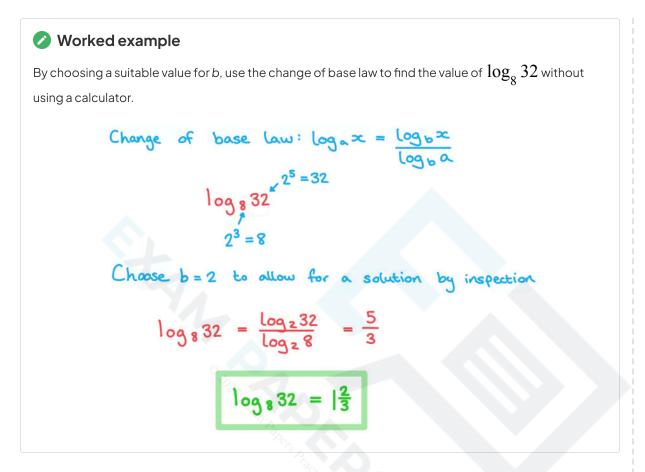
#### How do I change the base of a logarithm?

• The formula for changing the base of a logarithm is

$$\log_a x = \frac{\log_b x}{\log_b a}$$

- This is in the formula booklet
- The value you choose for b does not matter, however if you do not have a calculator, you can choose b such that the problem will be possible to solve







### 1.2.3 Solving Exponential Equations

### **Solving Exponential Equations**

#### What are exponential equations?

- An exponential equation is an equation where the unknown is a power
  - In simple cases the solution can be spotted without the use of a calculator
  - For example,

$$5^{2x} = 125$$
$$2x = 3$$
$$x = \frac{3}{2}$$

- In more complicated cases the laws of logarithms should be used to solve exponential equations
- The change of base law can be used to solve some exponential equations without a calculator
  - For example,

$$27^{x} = 9$$
$$x = \log_{27}9$$
$$= \frac{\log_{3}9}{\log_{3}27}$$
$$= \frac{2}{3}$$

#### How do we use logarithms to solve exponential equations?

- An exponential equation can be solved by taking logarithms of both sides
- The laws of indices may be needed to rewrite the equation first
- The laws of logarithms can then be used to solve the equation
  - In (log<sub>e</sub>) is often used
  - The answer is often written in terms of In
- A question my ask you to give your answer in a particular form
- Follow these steps to solve exponential equations
  - STEP 1: Take logarithms of both sides
  - STEP 2: Use the laws of logarithms to remove the powers
  - STEP 3: Rearrange to isolate x
  - STEP 4: Use logarithms to solve for x

#### What about hidden quadratics?



- Look for hidden squared terms that could be changed to form a quadratic
  - In particular look out for terms such as
    - $4^x = (2^2)^x = 2^{2x} = (2^x)^2$
    - $e^{2x} = (e^2)^x = (e^x)^2$



# Worked example Solve the equation $4^x - 3(2^{x+1}) + 9 = 0$ . Give your answer correct to three significant figures. Spot the hidden quadratic: $4^{\infty} = (2^{2})^{\infty} = (2^{\infty})^{2}$ By the laws of indices $2^{\infty+1} = 2^{\infty} \times 2^{1}$ $(2^{\infty})^2 - 3(2^{\infty+1}) + 9 = 0$ $= 2 \times 2^{\infty}$ $(2^{\infty})^2 - 3 \times 2 \times 2^{\infty} + 9 = 0$ $(2^{\infty})^2 - 6 \times 2^{\infty} + 9 = 0$ Let $u = 2^{\infty} u^2 - 6u + 9 = 0$ (u - 3)(u - 3) = 0 $u = 3 \therefore 2^{\infty} = 3$ Solve the exponential equation $2^{\infty} = 3$ Step 1: Take Logarithms of both sides : $\ln(2^{x}) = \ln(3)$ Step 2: Use the law $\log_a x^m = m \log_a x \quad x \ln 2 = \ln 3$ Step 3: Rearrange to isolate or $\infty = \frac{\ln 3}{\ln 2}$ Step 4: Solve $x = \frac{\ln 3}{\ln 2} = 1.584...$ $\infty = 1.58$ (3s.f.)