DP IB Maths: AA SL

1.2 Exponentials & Logs

Contents

1.2.1 Introduction to Logarithms

Introduction to Logarithms

What are logarithms?

- A logarithm is the inverse of an exponent
  - If $a^x = b$ then $\log_a(b) = x$ where $a > 0$, $b > 0$, $a \neq 1$
    - This is in the formula booklet
    - The number *a* is called the **base** of the logarithm
    - Your GDC will be able to use this function to solve equations involving exponents
- Try to get used to 'reading' logarithm statements to yourself
  - $\log_a(b) = x$ would be read as "the power that you raise a to, to get b, is x"
  - So $\log_5 125 = 3$ would be read as "the power that you raise 5 to, to get 125, is 3"
- Two important cases are:
  - $\ln x = \log_e(x)$
    - Where e is the mathematical constant 2.718…
    - This is called the **natural logarithm** and will have its own button on your GDC
  - $\log x = \log_{10}(x)$
    - Logarithms of **base 10** are used often and so abbreviated to **log *x***

Why use logarithms?

- Logarithms allow us to solve equations where the exponent is the unknown value
  - We can solve some of these by inspection
    - For example, for the equation $2^x = 8$ we know that *x* must be 3
  - Logarithms allow use to solve more complicated problems
    - For example, the equation $2^x = 10$ does not have a clear answer
    - Instead, we can use our GDCs to find the value of $\log_2 10$

Worked example

Solve the following equations:

i) $x = \log_3 27$,

$x = \log_3 27 \iff 3^x = 27$

We can see from inspection:

$3^3 = 27 \iff x = 3$

$\boxed{x = 3}$

OR: Use GDC to find answer directly.

ii) $2^x = 21.4$, giving your answer to 3 s.f.

$2^x = 21.4$ This cannot be seen from inspection:

$2^x = 21.4 \iff x = \log_2 21.4$

Use GDC to find answer directly.

$\log_2 21.4 = 4.4195...$

$\boxed{x = 4.42 \text{ (3 s.f.)}}$

1.2.2 Laws of Logarithms

Laws of Logarithms

What are the laws of logarithms?

- Laws of logarithms allow you to simplify and manipulate expressions involving logarithms
  - The laws of logarithms are equivalent to the **laws of indices**
- The laws you need to know are, given $a,\ x,\ y\ >\ 0$:
  - $\log_a xy = \log_a x + \log_a y$
    - This relates to $a^x \times a^y = a^{x+y}$
  - $\log_a \frac{x}{y} = \log_a x - \log_a y$
    - This relates to $a^x \div a^y = a^{x-y}$
  - $\log_a x^m = m\log_a x$
    - This relates to $(a^x)^y = a^{xy}$
- These laws are **in the formula booklet** so you do not need to remember them
  - You must make sure you know how to use them

Useful results from the laws of logarithms

- Given $a\ >\ 0\ ,\ a\ \neq\ 1$
  - $\log_a 1 = 0$
    - This is equivalent to $a^0 = 1$
- If we substitute *b* for *a* into the given identity in the formula booklet
  - $a^x = b \Leftrightarrow \log_a b = x$ where $a\ >\ 0,\ b\ >\ 0,\ a\ \neq\ 1$
  - $a^x = a \Leftrightarrow \log_a a = x$ gives $a^1 = a \Leftrightarrow \log_a a = 1$
    - This is an important and useful result
- Substituting this into the third law gives the result
  - $\log_a a^k = k$
- Taking the inverse of its operation gives the result
  - $a^{\log_a x} = x$
- From the third law we can also conclude that
  - $\log_a \frac{1}{x} = -\log_a x$

- These useful results are **not in the formula booklet** but can be deduced from the laws that are
- Beware...
  - ...$\log_a(x+y) \neq \log_a x + \log_a y$
- These results apply to $\ln x$ $(\log_e x)$ too
  - Two particularly useful results are
    - $\ln e^x = x$
    - $e^{\ln x} = x$
- Laws of logarithms can be used to ...
  - simplify expressions
  - solve logarithmic equations
  - solve exponential equations

Worked example

a) Write the expression $2\log 4 - \log 2$ in the form $\log k$, where $k \in \mathbb{Z}$.

Using the law $\log_a x^m = m\log_a x$

$2\log 4 = \log 4^2 = \log 16$

$2\log 4 - \log 2 = \log 4^2 - \log 2$

$= \log 16 - \log 2$

Using the law $\log_a \frac{x}{y} = \log_a x - \log_a y$

$\log 16 - \log 2 = \log \frac{16}{2} = \log 8$

$\boxed{2\log 4 - \log 2 = \log 8}$

b) Hence, or otherwise, solve $2\log 4 - \log 2 = -\log \frac{1}{x}$.

To solve $2\log 4 - \log 2 = \log\frac{1}{x}$ rewrite as

$$\log 8 = -\log\frac{1}{x}$$

from part (a)

Use the index law $\frac{1}{x} = x^{-1}$

$$\log 8 = -\log x^{-1}$$

$$\log 8 = \log x$$

$\log_a x^m = m\log_a x$

$$8 = x$$

$$\boxed{x = 8}$$

Change of Base

Why change the base of a logarithm?

- The laws of logarithms can only be used if the logs have the same **base**
  - If a problem involves logarithms with different bases, you can change the base of the logarithm and then apply the laws of logarithms
- **Changing the base** of a logarithm can be particularly useful if you need to evaluate a log problem **without a calculator**
  - Choose the base such that you would know how to solve the problem from the equivalent exponent

How do I change the base of a logarithm?

- The formula for changing the base of a logarithm is

$$\log_a x = \frac{\log_b x}{\log_b a}$$

- This is **in the formula booklet**
- The value you choose for *b* does not matter, however if you do not have a calculator, you can choose *b* such that the problem will be possible to solve

Worked example

By choosing a suitable value for *b*, use the change of base law to find the value of $\log_8 32$ without using a calculator.

Change of base law: $\log_a x = \dfrac{\log_b x}{\log_b a}$

$2^5 = 32$

$\log_8 32$

$2^3 = 8$

Choose $b = 2$ to allow for a solution by inspection

$$\log_8 32 = \frac{\log_2 32}{\log_2 8} = \frac{5}{3}$$

$$\boxed{\log_8 32 = 1\tfrac{2}{3}}$$

1.2.3 Solving Exponential Equations

Solving Exponential Equations

What are exponential equations?

- An exponential equation is an equation where the unknown is a power
  - In simple cases the solution can be spotted without the use of a calculator
  - For example,

$$5^{2x} = 125$$
$$2x = 3$$
$$x = \frac{3}{2}$$

- In more complicated cases the laws of logarithms should be used to solve exponential equations
- The **change of base** law can be used to solve some exponential equations without a calculator
  - For example,

$$27^x = 9$$
$$x = \log_{27}9$$
$$= \frac{\log_3 9}{\log_3 27}$$
$$= \frac{2}{3}$$

How do we use logarithms to solve exponential equations?

- An exponential equation can be solved by taking logarithms of both sides
- The **laws of indices** may be needed to rewrite the equation first
- The **laws of logarithms** can then be used to solve the equation
  - **ln ($\log_e$)** is often used
  - The answer is often written in terms of ln
- A question my ask you to give your answer in a particular form
- Follow these steps to solve exponential equations
  - STEP 1: Take logarithms of both sides
  - STEP 2: Use the laws of logarithms to remove the powers
  - STEP 3: Rearrange to isolate *x*
  - STEP 4: Use logarithms to solve for *x*

What about hidden quadratics?

- Look for hidden squared terms that could be changed to form a quadratic
  - In particular look out for terms such as
    - $4^x = (2^2)^x = 2^{2x} = (2^x)^2$
    - $e^{2x} = (e^2)^x = (e^x)^2$

Worked example

Solve the equation $4^x - 3(2^{x+1}) + 9 = 0$. Give your answer correct to three significant figures.

Spot the hidden quadratic: $4^x = (2^2)^x = (2^x)^2$

By the laws of indices $2^{x+1} = 2^x \times 2^1$

$= 2 \times 2^x$

$(2^x)^2 - 3(2^{x+1}) + 9 = 0$

$(2^x)^2 - 3 \times 2 \times 2^x + 9 = 0$

$(2^x)^2 - 6 \times 2^x + 9 = 0$

Let $u = 2^x$ $\quad u^2 - 6u + 9 = 0$

$(u-3)(u-3) = 0$

$u = 3 \quad \therefore \quad 2^x = 3$

Solve the exponential equation $2^x = 3$

Step 1: Take logarithms of both sides: $\ln(2^x) = \ln(3)$

Step 2: Use the law $\log_a x^m = m\log_a x$ $\quad x\ln 2 = \ln 3$

Step 3: Rearrange to isolate x $\quad x = \frac{\ln 3}{\ln 2}$

Step 4: Solve $\quad x = \frac{\ln 3}{\ln 2} = 1.584\ldots$

$$\boxed{x = 1.58 \text{ (3s.f.)}}$$