DP IB Maths: AA HL

1.10 Systems of Linear Equations

Contents

1.10.1 Systems of Linear Equations

Introduction to Systems of Linear Equations

What are systems of linear equations?

- A linear equation is an equation of the first order (**degree 1**)
  - This means that the **maximum degree** of each term is 1
  - These are examples of linear equations:
    - $2x + 3y = 5$ & $5x - y = 10 + 5z$
  - These are examples of non-linear equations:
    - $x^2 + 5x + 3 = 0$ & $3x + 2xy - 5y = 0$
    - The terms x^2 and xy have degree 2
- A system of linear equations is where **two or more linear equations** involve the **same variables**
  - These are also called **simultaneous equations**
- If there are ***n*** **variables** then you will need **at least** ***n*** **equations** in order to solve it
  - For your exam n will be 2 or 3
- A **2×2 system** of linear equations can be written as
  - $$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$
- A **3×3 system** of linear equations can be written as
  - $$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

What do systems of linear equations represent?

- The most common application of systems of linear equations is in **geometry**
- For a **2×2** system
  - Each equation will represent a **straight line in 2D**
  - The solution (if it exists and is unique) will correspond to the **coordinates** of the point where the **two lines intersect**
- For a **3×3** system
  - Each equation will represent a **plane in 3D**
  - The solution (if it exists and is unique) will correspond to the **coordinates** of the point where the **three planes intersect**

Systems of Linear Equations

How do I set up a system of linear equations?

- Not all questions will have the equations written out for you
- There will be **bits of information** given about the variables
  - **Two bits** of information for a **2×2 system**
  - **Three bits** of information for a **3×3 system**
  - Look out for clues such as 'assuming a linear relationship'
- Choose to assign ***x*, *y* & *z*** to the given variables
  - This will be helpful if using a GDC to solve
- Or you can choose to use more meaningful variables if you prefer
  - Such as c for the number of cats and d for the number of dogs

How do I use my GDC to solve a system of linear equations?

- You can use your **GDC to solve** the system on the **calculator papers (paper 2 & paper 3)**
- Your GDC will have a function within the algebra menu to solve a system of linear equations
- You will need to choose the number of equations
  - For two equations the variables will be x and y
  - For three equations the variables will be x, y and z
- If required, write the equations in the given form
  - $ax + by = c$
  - $ax + by + cz = d$
- Your GDC will display the values of x and y (or x, y, and z)

Worked example

On a mobile phone game, a player can purchase one of three power-ups (fire, ice, electricity) using their points.

- Adam buys 5 fire, 3 ice and 2 electricity power-ups costing a total of 1275 points.
- Alice buys 2 fire, 1 ice and 7 electricity power-ups costing a total of 1795 points.
- Alex buys 1 fire and 1 ice power-ups which in total costs 5 points less than a single electricity power up.

Find the cost of each power-up.

Let x be the cost of a fire power-up
Let y be the cost of an ice power-up
Let z be the cost of an electricity power-up

Form 3 equations

$5x + 3y + 2z = 1275$

$2x + y + 7z = 1795$

$x + y = z - 5 \quad \longrightarrow \quad x + y - z = -5$

Write in form $ax + by + cz = d$

Type the 3 equations into the GDC and solve

$x = 120, \; y = 85, \; z = 210$

Fire costs 120 points
Ice costs 85 points
Electricity costs 210 points

1.10.2 Algebraic Solutions

Row Reduction

How can I write a system of linear equations?

- To save space we can just write the **coefficients without the variables**
- For 2 variables: $\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$ can be written shorthand as $\left[\begin{array}{cc|c} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{array}\right]$
- For 3 variables: $\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$ can be written shorthand as $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$

What is a row reduced system of linear equations?

- A system of linear equations is in row reduced form if it is written as:
  - $\left[\begin{array}{ccc|c} A_1 & B_1 & C_1 & D_1 \\ 0 & B_2 & C_2 & D_2 \\ 0 & 0 & C_3 & D_3 \end{array}\right]$ which corresponds to $\begin{aligned} A_1x + B_1y + C_1z &= D_1 \\ B_2y + C_2z &= D_2 \\ C_3z &= D_3 \end{aligned}$
    - It is very helpful if the values of A_1, B_2, C_3 are **equal to 1**

What are row operations?

- **Row operations** are used to make the linear equations **simpler to solve**
  - They **do not affect the solution**
- You can **switch any two rows**
  - $\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$ can be written as $\left[\begin{array}{ccc|c} a_3 & b_3 & c_3 & d_3 \\ a_2 & b_2 & c_2 & d_2 \\ a_1 & b_1 & c_1 & d_1 \end{array}\right]$ using $r_1 \leftrightarrow r_3$
    - This is useful for getting zeros to the bottom
    - Or getting a one to the top
- You can **multiply any row by a (non-zero) constant**

- $$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$ can be written as $$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ ka_2 & kb_2 & kc_2 & kd_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$ using $k \times r_2 \rightarrow r_2$
  - This is useful for getting a 1 as the first non-zero value in a row
- You can **add multiples of a row to another row**
  - $$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$ can be written as $$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2+5a_3 & b_2+5b_3 & c_2+5c_3 & d_2+5d_3 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right]$$ using $r_2 + 5r_3 \rightarrow r_2$
    - This is useful for creating zeros under a 1

How can I row reduce a system of linear equations?

- **STEP 1**: Get a **1 in the top left corner**
  - You can do this by **dividing the row** by the current value in its place
  - If the current value is 0 or an awkward number then you can **swap rows first**
    - $$\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ * & * & * & * \\ * & * & * & * \end{array}\right]$$
- **STEP 2**: Get **0's in the entries below the 1**
  - You can do this by **adding/subtracting a multiple of the first row** to each row
    - $$\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ 0 & * & * & * \\ 0 & * & * & * \end{array}\right]$$
- **STEP 3**: **Ignore the first row and column** as they are now complete
  - **Repeat STEPS 1 – 2** to the remaining section
    - Get a 1: $$\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ 0 & 1 & C_2 & D_2 \\ 0 & * & * & * \end{array}\right]$$
    - Then 0 underneath: $$\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ 0 & 1 & C_2 & D_2 \\ 0 & 0 & * & * \end{array}\right]$$
- **STEP 4**: Get a **1 in the third row**
  - Using the same idea as **STEP 1**
    - $$\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ 0 & 1 & C_2 & D_2 \\ 0 & 0 & 1 & D_3 \end{array}\right]$$

How do I solve a system of linear equations once it is in row reduced form?

- Once you row reduced the equations you can then **convert back to the variables**
  - $\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ 0 & 1 & C_2 & D_2 \\ 0 & 0 & 1 & D_3 \end{array}\right]$ corresponds to $\begin{aligned} x + B_1 y + C_1 z &= D_1 \\ y + C_2 z &= D_2 \\ z &= D_3 \end{aligned}$
- Solve the equations **starting at the bottom**
  - You have the value for z from the third equation
  - Substitute z into the second equation and solve for y
  - Substitute z and y into the first each and solve for x

Worked example

Solve the following system of linear equations using algebra.

$$2x-3y+4z=14$$
$$x+2y-2z=-2$$
$$3x-y-2z=10$$

Write without the variables

$$\left[\begin{array}{ccc|c} 2 & -3 & 4 & 14 \\ 1 & 2 & -2 & -2 \\ 3 & -1 & -2 & 10 \end{array}\right]$$

Swap rows to get 1 in top left corner

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ 2 & -3 & 4 & 14 \\ 3 & -1 & -2 & 10 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

Add multiples of R_1 to R_2 and R_3 to get zeros under the 1

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ 0 & -7 & 8 & 18 \\ 0 & -7 & 4 & 16 \end{array}\right] \begin{array}{l} \\ R_2-2R_1 \rightarrow R_2 \\ R_3-3R_1 \rightarrow R_3 \end{array}$$

Multiple the second row to get a 1

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ 0 & 1 & -\frac{8}{7} & -\frac{18}{7} \\ 0 & -7 & 4 & 16 \end{array}\right] \begin{array}{l} \\ R_2 \times -\frac{1}{7} \rightarrow R_2 \\ \\ \end{array}$$

Repeat the steps

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ 0 & 1 & -\frac{8}{7} & -\frac{18}{7} \\ 0 & 0 & -4 & -2 \end{array}\right] R_3+7R_2 \rightarrow R_3 \qquad \left[\begin{array}{ccc|c} 1 & 2 & -2 & -2 \\ 0 & 1 & -\frac{8}{7} & -\frac{18}{7} \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right] R_3 \times -\frac{1}{4} \rightarrow R_3$$

Write out the equations starting at the bottom

$z=\frac{1}{2}$

$y-\frac{8}{7}z=-\frac{18}{7} \Rightarrow y-\frac{4}{7}=-\frac{18}{7} \Rightarrow y=-\frac{14}{7}=-2$

$x+2y-2z=-2 \Rightarrow x-4-1=-2 \Rightarrow x=3$

$\boxed{x=3,\quad y=-2,\quad z=\frac{1}{2}}$

Number of Solutions to a System

How many solutions can a system of linear equations have?

- There could be
  - 1 **unique solution**
  - **No solutions**
  - An **infinite number** of solutions
- You can determine the case by looking at the row reduced form

How do I know if the system of linear equations has no solutions?

- Systems with **no solutions** are called **inconsistent**
- When trying to solve the system after using the row reduction method you will end up with a **mathematical statement which is never true**:
  - Such as: 0 = 1
- The **row reduced system will contain**:
  - **At least one row** where the entries to the **left of the line are zero** and the entry on the **right of the line is non-zero**
    - Such a row is called **inconsistent**
  - For example:
    - Row 2 is inconsistent $\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ 0 & 0 & 0 & D_2 \\ 0 & 0 & 1 & D_3 \end{array}\right]$ if D_2 is non-zero

How do I know if the system of linear equations has an infinite number of solutions?

- Systems with **at least one solution** are called **consistent**
  - The solution could be unique or there could be an infinite number of solutions
- When trying to solve the system after using the row reduction method you will end up with a **mathematical statement which is always true**
  - Such as: 0 = 0
- The **row reduced system will contain**:
  - **At least one row** where **all the entries are zero**
  - **No inconsistent rows**
  - For example:
    - $\left[\begin{array}{ccc|c} 1 & B_1 & C_1 & D_1 \\ 0 & 1 & C_2 & D_2 \\ 0 & 0 & 0 & 0 \end{array}\right]$

How do I find the general solution of a dependent system?

- A **dependent system** of linear equations is one where there are **infinite number of solutions**

- The general solution will depend on **one or two parameters**
- In the case where **two rows are zero**
    - Let the **variables corresponding** to the **zero rows** be equal to the **parameters** λ & μ
        - For example: If the first and second rows are zero rows then let $x = \lambda$ & $y = \mu$
    - Find the **third** variable in terms of the two parameters using the equation from the third row
        - For example: $z = 4\lambda - 5\mu + 6$
- In the case where **only one row is zero**
    - Let the **variable corresponding** to the **zero row** be equal to the **parameter** λ
        - For example: If the first row is a zero row then let $x = \lambda$
    - Find the **remaining two variables in terms of the parameter** using the equations formed by the other two rows
        - For example: $y = 3\lambda - 5$ & $z = 7 - 2\lambda$

Worked example

$$x + 2y - z = 3$$
$$3x + 7y + z = 4$$
$$x - 9z = k$$

a) Given that the system of linear equations has an infinite number of equations, find the value of k.

Write without the variables

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 3 & 7 & 1 & 4 \\ 1 & 0 & -9 & k \end{array}\right]$$

Use the row reduction method

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & k-3 \end{array}\right] \begin{array}{l} \\ r_2 - 3r_1 \rightarrow r_2 \\ r_3 - r_1 \rightarrow r_3 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & k-13 \end{array}\right] \begin{array}{l} \\ \\ r_3 + 2r_2 \rightarrow r_3 \end{array}$$

There are an infinite number of solutions if a row is zero

$k - 13 = 0$

$\boxed{k = 13}$

b) Find a general solution to the system.

The third row is zero so let the third variable (z) equal a parameter

$z = \lambda$

Use equations to find expressions for the other variables

$y + 4z = -5 \Rightarrow y + 4\lambda = -5 \Rightarrow y = -4\lambda - 5$

$x + 2y - 1 = 3 \Rightarrow x - 8\lambda - 10 - \lambda = 3 \Rightarrow x = 9\lambda + 13$

$\boxed{x = 9\lambda + 13,\ y = -4\lambda - 5,\ z = \lambda \text{ for } \lambda \in \mathbb{R}}$