DP IB Maths: AI SL

1.1 Number Toolkit

Contents

1.1.1 Standard Form

Standard Form

Standard form (sometimes called **scientific notation** or **standard index form**) gives us a way of writing very big and very small numbers using powers of 10.

Why use standard form?

- Some numbers are too big or too small to write easily or for your calculator to display at all
  - Imagine the number 50^{50}, the answer would take 84 digits to write out
  - Try typing 50^{50} into your calculator, you will see it displayed in **standard form**
- Writing very big or very small numbers in standard form allows us to:
  - Write them more neatly
  - Compare them more easily
  - Carry out calculations more easily
- Exam questions could ask for your answer to be written in standard form

How is standard form written?

- In standard form numbers are always written in the form $a \times 10^k$ where a and k satisfy the following conditions:
  - $1 \leq a < 10$
    - So there is one non – zero digit before the decimal point
  - $k \in \mathbb{Z}$
    - So k must be an integer
  - $k > 0$ for large numbers
    - How many times a is multiplied by 10
  - $k < 0$ for small numbers
    - How many times a is divided by 10

How are calculations carried out with standard form?

- Your GDC will display large and small numbers in standard form when it is in normal mode
  - Your GDC may display standard form as aEn
    - For example, 2.1×10^{-5} will be displayed as $2.1\text{E} - 5$
    - If so, be careful to **rewrite the answer given in the correct form**, you will not get marks for copying directly from your GDC
- Your GDC will be able to carry out calculations in standard form
  - If you put your GDC into scientific mode it will automatically convert numbers into standard form
    - Beware that your GDC may have more than one mode when in scientific mode

- This relates to the number of significant figures the answer will be displayed in
- Your GDC may add extra zeros to fill spaces if working with a high number of significant figures, you do not need to write these in your answer

- To add or subtract numbers written in the form $a \times 10^k$ without your GDC you will need to write them in full form first
  - Alternatively you can use 'matching powers of 10', because if the powers of 10 are the same, then the 'number parts' at the start can just be added or subtracted normally
    - For example
    $$(6.3 \times 10^{14}) + (4.9 \times 10^{13}) = (6.3 \times 10^{14}) + (0.49 \times 10^{14}) = 6.79 \times 10^{14}$$
    - Or
    $$(7.93 \times 10^{-11}) - (5.2 \times 10^{-12}) = (7.93 \times 10^{-11}) - (0.52 \times 10^{-11}) = 7.41 \times 10^{-11}$$
- To multiply or divide numbers written in the form $a \times 10^k$ without your GDC you can either write them in full form first or use the laws of indices

✏️ Worked example

Calculate the following, giving your answer in the form $a \times 10^k$, where $1 \leq a < 10$ and $k \in \mathbb{Z}$.

i) 3780×200

Using GDC: Choose scientific mode.
Input directly into GDC as ordinary numbers.

$3780 \times 200 = 7.56 \times 10^5$

GDC will automatically give answer in standard form.

Without GDC:

Calculate the value:

$3780 \times 200 = 756000$

Convert to standard form:

$756000 = 7.56 \times 10^5$

$$\boxed{7.56 \times 10^5}$$

ii) $(7 \times 10^5) - (5 \times 10^4)$

Using GDC: Choose scientific mode.
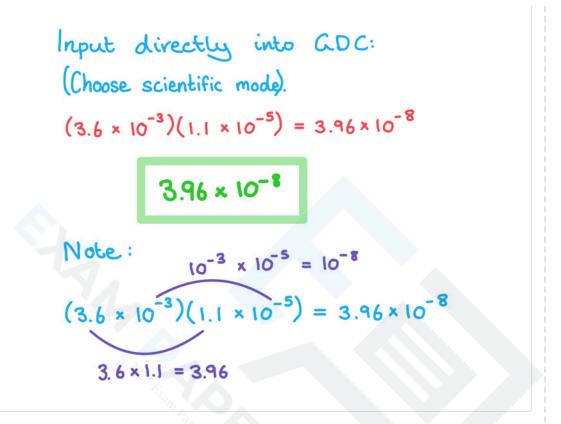
Input directly into GDC

$$7 \times 10^5 - 5 \times 10^4 = 6.5 \times 10^5$$

This may be displayed as 6.5E5

Without GDC:

Convert to ordinary numbers:

$$7 \times 10^5 = 700\,000$$

$$5 \times 10^4 = 50\,000$$

Carry out the calculation:

$$700\,000 - 50000 = 650000$$

Convert to standard form:

$$650000 = 6.5 \times 10^5$$

$$\boxed{6.5 \times 10^5}$$

iii) $(3.6 \times 10^{-3})(1.1 \times 10^{-5})$

Input directly into GDC:
(Choose scientific mode).

$(3.6 \times 10^{-3})(1.1 \times 10^{-5}) = 3.96 \times 10^{-8}$

$$3.96 \times 10^{-8}$$

Note:

$10^{-3} \times 10^{-5} = 10^{-8}$

$(3.6 \times 10^{-3})(1.1 \times 10^{-5}) = 3.96 \times 10^{-8}$

$3.6 \times 1.1 = 3.96$

1.1.2 Exponents & Logarithms
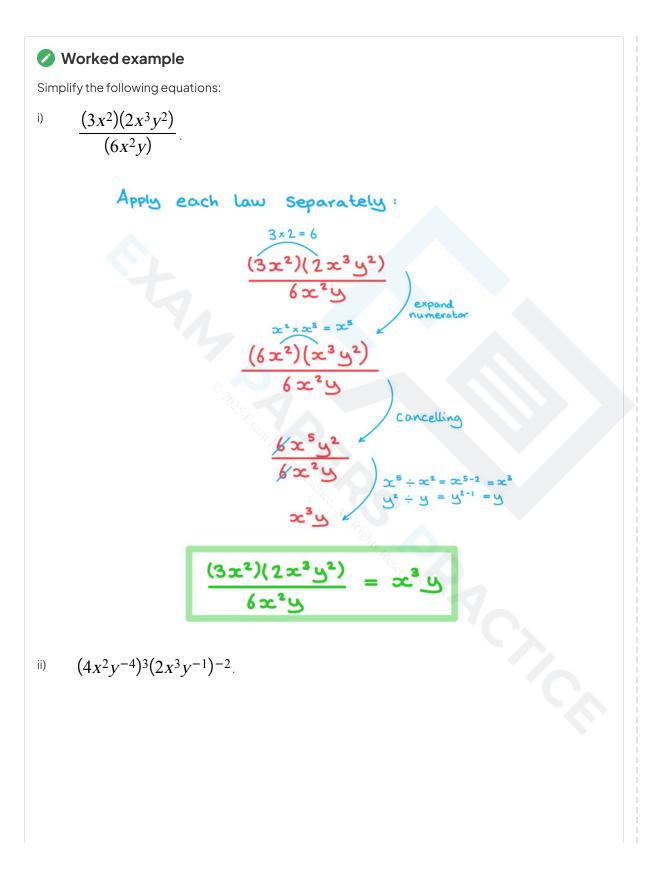
Laws of Indices

What are the laws of indices?

- Laws of indices (or index laws) allow you to simplify and manipulate expressions involving exponents
  - An exponent is a power that a number (called the base) is raised to
  - Laws of indices can be used when the numbers are written with the same base
- The index laws you need to know are:

  - $(xy)^m = x^m y^m$

  - $\left(\dfrac{x}{y}\right)^m = \dfrac{x^m}{y^m}$

  - $x^m \times x^n = x^{m+n}$

  - $x^m \div x^n = x^{m-n}$

  - $(x^m)^n = x^{mn}$

  - $x^1 = x$

  - $x^0 = 1$

  - $\dfrac{1}{x^m} = x^{-m}$

- These laws are **not in the formula booklet** so you must remember them
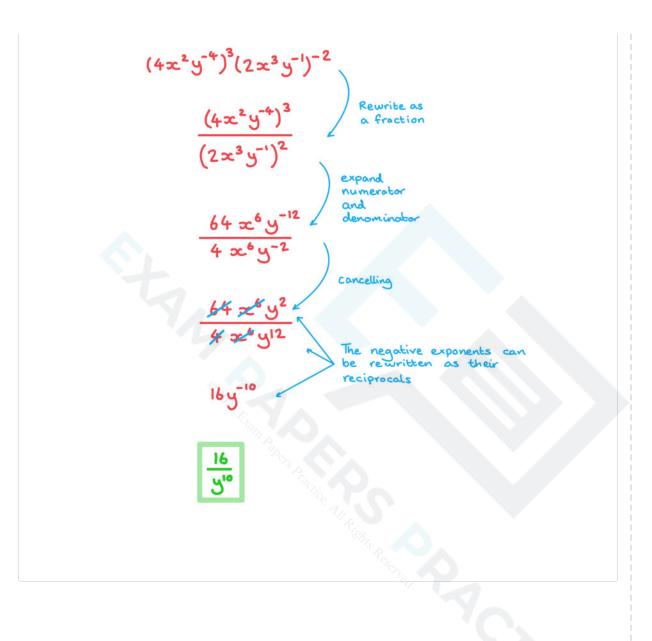
How are laws of indices used?

- You will need to be able to carry out multiple calculations with the laws of indices
  - Take your time and apply each law individually
  - Work with numbers first and then with algebra
- Index laws only work with terms that have the same base, make sure you **change the base** of the term before using any of the index laws
  - Changing the base means rewriting the number as an exponent with the base you need
  - For example, $9^4 = (3^2)^4 = 3^{2 \times 4} = 3^8$
  - Using the above can them help with problems like $9^4 \div 3^7 = 3^8 \div 3^7 = 3^1 = 3$

✏️ Worked example

Simplify the following equations:

i) $\dfrac{(3x^2)(2x^3y^2)}{(6x^2y)}$.

Apply each law separately:

$$3 \times 2 = 6$$

$$\frac{(3x^2)(2x^3y^2)}{6x^2y}$$

expand numerator

$$x^2 \times x^3 = x^5$$

$$\frac{(6x^2)(x^3y^2)}{6x^2y}$$

cancelling

$$\frac{6x^5y^2}{6x^2y}$$

$$x^5 \div x^2 = x^{5-2} = x^3$$
$$y^2 \div y = y^{2-1} = y$$

$$x^3y$$

$$\boxed{\frac{(3x^2)(2x^3y^2)}{6x^2y} = x^3y}$$

ii) $(4x^2y^{-4})^3(2x^3y^{-1})^{-2}$.

$$(4x^2y^{-4})^3(2x^3y^{-1})^{-2}$$

$$\frac{(4x^2y^{-4})^3}{(2x^3y^{-1})^2}$$

Rewrite as a fraction

$$\frac{64x^6y^{-12}}{4x^6y^{-2}}$$

expand numerator and denominator

$$\frac{64\,\cancel{x^6}\,y^2}{4\,\cancel{x^6}\,y^{12}}$$

Cancelling

The negative exponents can be rewritten as their reciprocals

$$16y^{-10}$$

$$\boxed{\frac{16}{y^{10}}}$$
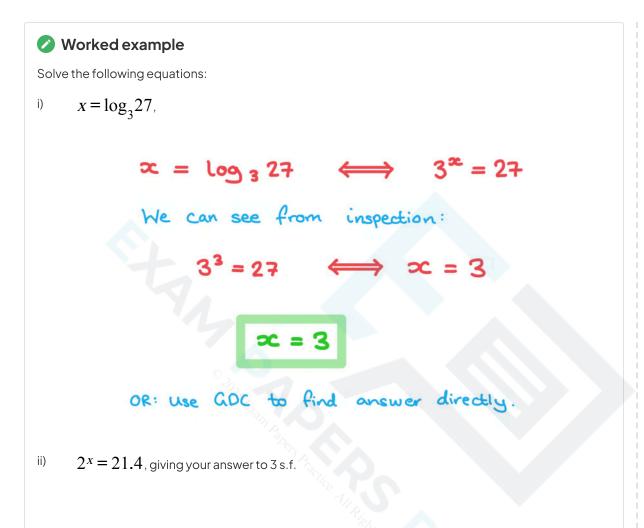
Introduction to Logarithms

What are logarithms?

- A logarithm is the inverse of an exponent
  - If $a^x = b$ then $\log_a(b) = x$ where $a > 0, b > 0, a \neq 1$
    - This is in the formula booklet
    - The number a is called the **base** of the logarithm
    - Your GDC will be able to use this function to solve equations involving exponents
- Try to get used to 'reading' logarithm statements to yourself
  - $\log_a(b) = x$ would be read as "the power that you raise a to, to get b, is x"
  - So $\log_5 125 = 3$ would be read as "the power that you raise 5 to, to get 125, is 3"
- Two important cases are:
  - $\ln x = \log_e(x)$
    - Where e is the mathematical constant 2.718...
    - This is called the **natural logarithm** and will have its own button on your GDC
  - $\log x = \log_{10}(x)$
    - Logarithms of **base 10** are used often and so abbreviated to **log x**

Why use logarithms?

- Logarithms allow us to solve equations where the exponent is the unknown value
  - We can solve some of these by inspection
    - For example, for the equation $2^x = 8$ we know that x must be 3
  - Logarithms allow use to solve more complicated problems
    - For example, the equation $2^x = 10$ does not have a clear answer
    - Instead, we can use our GDCs to find the value of $\log_2 10$

✏️ Worked example

Solve the following equations:

i) $x = \log_3 27$,

$$x = \log_3 27 \quad \Longleftrightarrow \quad 3^x = 27$$

We can see from inspection:

$$3^3 = 27 \quad \Longleftrightarrow \quad x = 3$$

$$\boxed{x = 3}$$

OR: Use GDC to find answer directly.

ii) $2^x = 21.4$, giving your answer to 3 s.f.

$2^x = 21.4$ This cannot be seen
                from inspection:

$2^x = 21.4 \iff x = \log_2 21.4$

Use GDC to find answer directly.

$\log_2 21.4 = 4.4195\ldots$

$$\boxed{x = 4.42 \ (3 \text{ s.f.})}$$

1.1.3 Approximation & Estimation

Upper & Lower Bounds

What are bounds?

- Bounds are the smallest **(lower bound, LB)** and largest **(upper bound, UB)** numbers that a **rounded number** can lie between
  - It simply means how low or high the number could have been before it was rounded
- The bounds for a number, X, can be written as $\mathrm{LB} \leq X < \mathrm{UB}$
  - Note that the lower bound is included in the range of values X could have taken but the upper bound is not

How do we find bounds?

- The basic rule is "half up, half down"
  - To find the upper bound add on half the degree of accuracy
  - To find the lower bound take off half the degree of accuracy
- Remember that the upper bound is the cut off point for the greatest value that the number could have been rounded from but will not actually round to the number itself

How do we calculate using bounds?

- Find bounds before carrying out the calculation and then use the rules:
  - To add **UB = UB + UB** and **LB = LB + LB**
  - To multiply **UB = UB × UB** and **LB = LB × LB**
  - To divide **UB = UB / LB** and **LB = LB / UB**
  - To subtract **UB = UB – LB** and **LB = LB – UB**
- Use logic to decide which bound to use within the calculation
  - For example if you are finding the **maximum** volume of a sphere with the radius given correct to 1 decimal place substitute the **upper bound** of the radius into your calculation for the volume
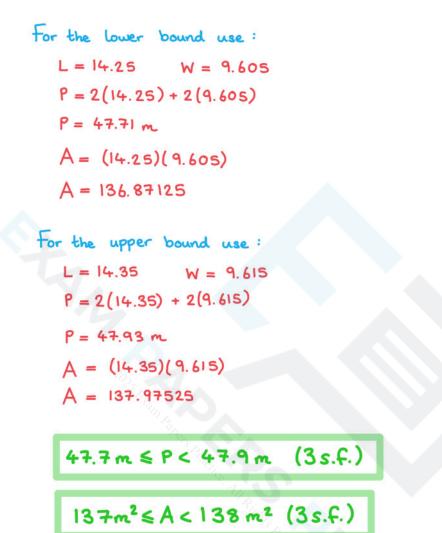
✏️ **Worked example**

A rectangular field has length, L, of 14.3 m correct to 1 decimal place and width, W, of 9.61 m correct to 2 decimal places.

a)   Calculate the lower and upper bound for L and W.

L = 14.3 m (1 d.p.) the degree of accuracy
                              is 1 d.p (0.1)
find half the degree of accuracy :

$$\frac{0.1}{2} = 0.05$$

The upper bound is
            14.3 + 0.05 = 14.35
The lower bound is
            14.3 − 0.05 = 14.25

$$14.25 \leqslant L < 14.35$$

W = 9.61 m (2 d.p.) the degree of accuracy
                                is 2 d.p (0.01)
find half the degree of accuracy :

$$\frac{0.01}{2} = 0.005$$

The upper bound is
            9.61 + 0.005 = 9.615
The lower bound is
            9.61 − 0.005 = 9.605

$$9.605 \leqslant W < 9.615$$

b)   Calculate the lower and upper bound for the perimeter, P, and area, A, of the field.

For the lower bound use :

$L = 14.25 \qquad W = 9.605$

$P = 2(14.25) + 2(9.605)$

$P = 47.71 \text{ m}$

$A = (14.25)(9.605)$

$A = 136.87125$

For the upper bound use :

$L = 14.35 \qquad W = 9.615$

$P = 2(14.35) + 2(9.615)$

$P = 47.93 \text{ m}$

$A = (14.35)(9.615)$

$A = 137.97525$

$$47.7 \text{ m} \leq P < 47.9 \text{ m} \quad (3\text{ s.f.})$$

$$137 \text{ m}^2 \leq A < 138 \text{ m}^2 \quad (3\text{ s.f.})$$
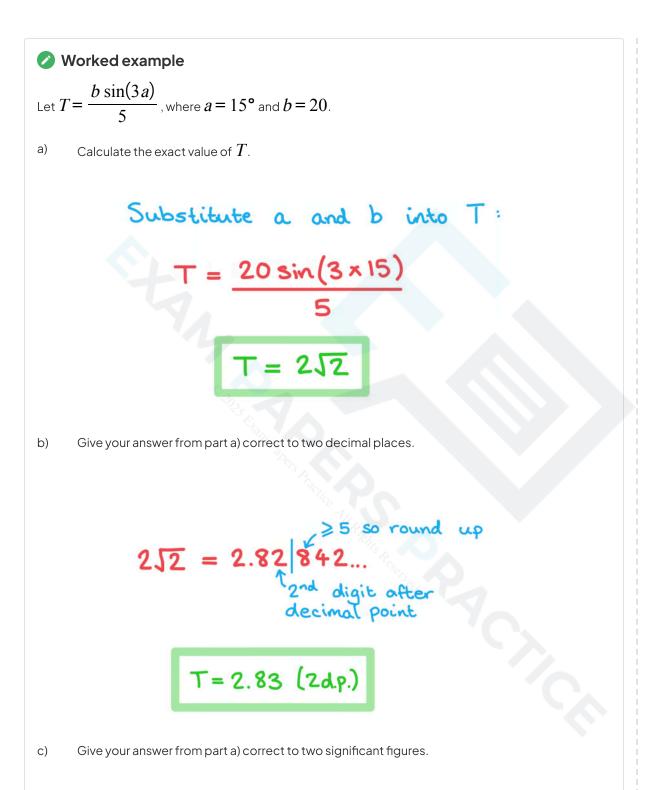
Approximating Values

How do I know what to round my answer to?

- Unless otherwise told, always round your answers to **3 significant figures** (3 s.f.)
    - The first non-zero digit is the first **significant** digit
    - The first digit after the third significant digit determines whether to 'round up' ($\geq$ 5) or 'leave it alone' (<5)
        - where the 'it' we are rounding up or leaving alone is the third significant figure
    - Your final answer will have three **significant digits** and the rest will be zero
        - Any zero **after** the first significant digit is still significant
        - For large numbers be careful not to change the **place value** of the significant digits, you will have to fill in any zeros after the third significant figure
        - If your GDC is in **scientific mode** it may display unnecessary zeros after the decimal point, you do not need to copy these
- Look out for any questions that ask you to round your answer in a different way
    - Questions often ask for **2 decimal places** (2 d.p.)
        - Your final answer will only have 2 digits after the decimal point
        - For 2 d.p. it is the third digit after the decimal place that determines whether to 'round up' (5) or 'leave it alone' (<5)
- If you are working with a **currency** you must choose the appropriate degree of accuracy
    - For most this will be a **whole number**
        - E.g. yen, yuan, peso
    - For others this will be to **2 decimal places**
        - E.g. dollars, euro, pounds
    - It will be clear from the question which currency you are using and how you should round your answer
        - The question will state the name of the currency and the symbol you should use as a unit
        - E.g. YEN, ¥

Are there cases when I always have to round up?

- Yes – there are cases when it makes sense to always round up (or down)
- These normally involve finding the **minimum** or **maximum number** of objects
    - For example consider the scenario: There are 26 people and 5 people can fit in a single vehicle, how many vehicles are needed?
        - $\dfrac{26}{5} = 5.2$ and normally we'd round to 5
        - However 5 vehicles wouldn't be enough as there would only be room for 25 people
        - In this case we would round up to find the **minimum** number needed
- These kind of problems can be solved by inequalities
    - For $x > k$ take the **smallest value** of x at the appropriate degree of accuracy that is **greater than** k
        - For example: Using 3sf the smallest solution to $x > 2.5731...$ is $x = 2.58$
    - For $x < k$ take the **biggest value** of x at the appropriate degree of accuracy that is **less than** k
        - For example: The biggest integer solution to $x < 10.901...$ is $x = 10$

Worked example

Let $T = \dfrac{b\sin(3a)}{5}$, where $a = 15°$ and $b = 20$.

a) Calculate the exact value of T.

Substitute a and b into T:

$$T = \frac{20\sin(3 \times 15)}{5}$$

$$\boxed{T = 2\sqrt{2}}$$

b) Give your answer from part a) correct to two decimal places.

$\geqslant 5$ so round up

$$2\sqrt{2} = 2.82|842...$$

2nd digit after decimal point

$$\boxed{T = 2.83 \ (2\text{d.p.})}$$

c) Give your answer from part a) correct to two significant figures.

first significant figure

<5 so don't round up

$2\sqrt{2} = 2.8|2842...$

2nd significant figure

$T = 2.8$ (2 s.f.)

Percentage Error

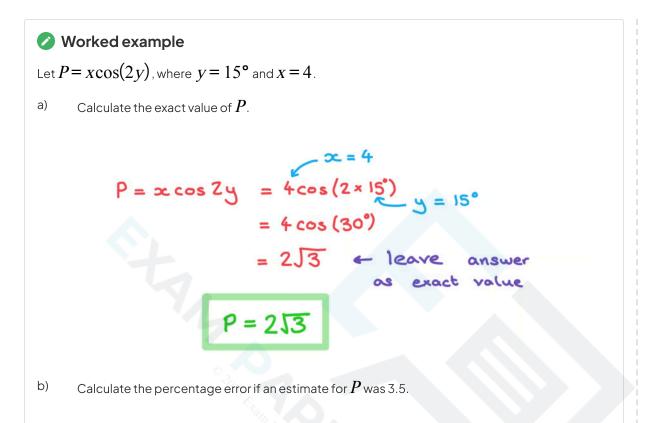## What is percentage error?

- Percentage error is how far away from the actual value an estimated or rounded answer is
  - Percentage error can be calculated using the formula

$$\varepsilon = \left| \frac{V_A - V_E}{V_E} \right| \times 100\%$$

  - where V_E is the exact value and V_A is the approximate value of V
    - The || is the **absolute value** meaning if you get a negative value within these straight brackets, you should take the **positive** value
    - This formula is **in the formula booklet** so you do not need to remember it
- The further away the estimated answer is from the true answer the greater the percentage error
- If the exact value is given as a surd or a multiple of π make sure you enter it into the formula exactly as you see it
- Percentage error should always be a positive number

✏️ **Worked example**

Let $P = x\cos(2y)$, where $y = 15°$ and $x = 4$.

a)   Calculate the exact value of P.

$$x = 4$$

$$P = x\cos 2y \quad = 4\cos(2 \times 15°)$$

$$y = 15°$$

$$= 4\cos(30°)$$

$$= 2\sqrt{3} \quad \leftarrow \text{leave answer as exact value}$$

$$\boxed{P = 2\sqrt{3}}$$

b)   Calculate the percentage error if an estimate for P was 3.5.

Percentage error formula:

$$\varepsilon = \left| \frac{V_A - V_E}{V_E} \right| \times 100\%$$

$V_A = 3.5$ (approximated value)

$V_E = 2\sqrt{3}$ (exact value)

Sub V_A and V_E into the formula:

$$\varepsilon = \left| \frac{3.5 - 2\sqrt{3}}{2\sqrt{3}} \right| \times 100\%$$

$$= 1.03629\ldots \%$$

$$\boxed{\varepsilon = 1.04\% \ (3\,\text{s.f.})}$$

Accuracy & Estimation

What are exact values?

- Exact values are forms that represent the full and precise value of a number
  - For example, π is an exact value and 3.14 is an approximation using 3 significant figures
- If a number has an infinite number of non-zero digits after the decimal point then you can use three dots to signal that the decimal representation goes on for example
  - For example, $\sqrt{2} = 1.414...$
- Exact values can involve

  - Fractions: $\dfrac{2}{7}$

  - Roots: $\sqrt{3}, \sqrt[5]{7}$

  - Logarithms: $\ln 2, \log_{10} 5$

  - Mathematical constants: π, e

- Your GDC might automatically give your answer as an exact answer
- If your GDC does not do this then you may need to evaluate parts of the expression separately and use algebra
  - For example: If $f(x) = e^x (2 + \sqrt{x})$ then your GDC will probably not give you the exact value of $f(2)$
  - You would insist evaluate it without a GDC to get the exact value: $f(2) = e^2 (2 + \sqrt{2})$

Why use estimation?

- We **estimate** to find approximate answers to difficult sums
- Or to check our answers are about the right size (order of magnitude)
  - For example, if the question is to find a length the answer cannot be negative
  - or if we are looking for the mean age of some people an answer of 150 must be incorrect
- Estimating an answer before carrying out a calculation will help you know what you are looking for and determine if your answer is likely to be correct or not
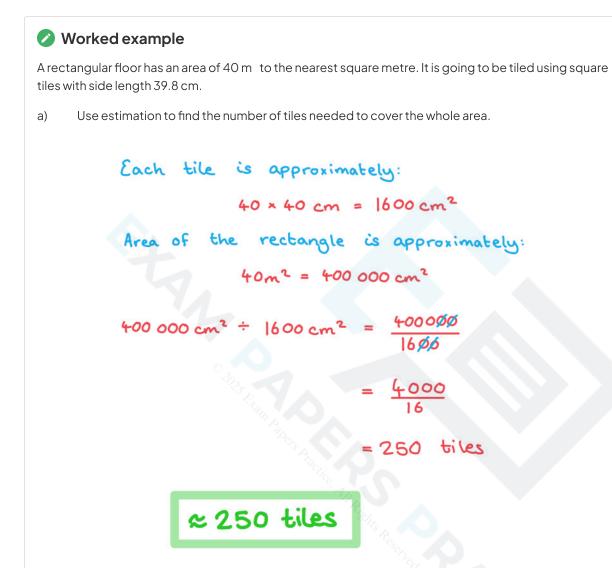- In real life estimation skills are used every day in many activities

How do I choose the correct answer?

- Sometimes a mathematical argument will lead to more than one answer
  - This is common with problems involving quadratics, you will usually have two solutions
  - If you have more than one solution after you have solved a problem, **always** check to see if they are both valid
- Most of the time you can simply use logic to choose the correct answer

- - If the problem involves length or area and one of the answers is negative, the true solution will be the positive answer
- Occasionally you will need to see if an answer can be valid
  - If one of your answers is $\cos x > 1$ for example, x will not give a true solution

✏️ **Worked example**

A rectangular floor has an area of 40 m to the nearest square metre. It is going to be tiled using square tiles with side length 39.8 cm.

a)   Use estimation to find the number of tiles needed to cover the whole area.

Each tile is approximately:

$$40 \times 40 \text{ cm} = 1600 \text{ cm}^2$$

Area of the rectangle is approximately:

$$40\text{m}^2 = 400\,000 \text{ cm}^2$$

$$400\,000 \text{ cm}^2 \div 1600 \text{ cm}^2 = \frac{400000}{1600}$$

$$= \frac{4000}{16}$$

$$= 250 \text{ tiles}$$

$$\approx 250 \text{ tiles}$$

b)   Given that there are 15 more tiles places length-wise than width-wise, find the approximate length and width of the floor.

Let the number of tiles covering the width of the floor be x, then the number of tiles covering the length will be $x + 15$.

number of tiles placed widthways ↘ number of tiles placed lengthways ↙

$$x(x + 15) = 250$$

$$x^2 + 15x - 250 = 0$$

$$x = 10 \quad \text{or} \quad x = -25$$

↑ not possible as x cannot be a negative number

Width of floor $\approx 10 \times 40$ cm

$$= 400 \text{ cm} = 4 \text{ m}$$

Length of floor $\approx 25 \times 40$ cm

$$= 1000 \text{ cm} = 10 \text{ m}$$

Length ≈ 10 m, Width ≈ 4 m

1.1.4 GDC: Solving Equations

Systems of Linear Equations

What are systems of linear equations?

- A linear equation is an equation of the first order (degree 1)
  - It is usually written in the form *ax + by + c = 0* where *a*, *b*, and *c* are constants
- A system of linear equations is where two or more linear equations work together
  - Usually there will be two equations with the variables *x* and *y*
  - The **variables** *x* and *y* will satisfy all equations
  - They are usually known as **simultaneous equations**
  - Occasionally there may be three equations with the variables *x*, *y* and *z*
- They can be complicated to solve but your GDC has a function allowing you to solve them
  - The question may say 'using technology, solve…'
    - This means you do not need to show a method of solving the system of equations, you can use your GDC

How do I use my GDC to solve a system of linear equations?

- Your GDC will have a function within the algebra menu to solve a system of linear equations
- You will need to choose the number of equations
  - For two equations the variables will be *x* and *y*
  - For three equations the variables will be *x*, *y* and *z*
- Enter the equations into your calculator as you see them written
- Your GDC will display the values of *x* and *y* (or *x*, *y*, and *z*)

How do I set up a system of linear equations?

- Not all questions will have the equations written out for you
- There will be two bits of information given about two variables
  - Look out for clues such as 'assuming a linear relationship'
- Choose to assign *x* to one of the given variables and *y* to the other
  - Or you can choose to use more meaningful variables if you prefer
  - Such as *c* for cats and *d* for dogs
- Write your system of equations in the form

$$ax + by = e$$
$$cx + dy = f$$

- Use your GDC to solve the system of equations
- This function on the GDC can also be used to find the points of intersection of two straight line graphs
  - You may wish to use the graphing section on your GDC to see the points of intersection

✏️ Worked example

A theme park has set ticket prices for adults and children. A group of three adults and nine children costs \$153 and a group of five adults and eleven children costs \$211.

i)   Set up a system of linear equations for the cost of adult and child tickets.

Set up variables :

Let the cost of an adult ticket be 'a'

Let the cost of a child ticket be 'c'

Set up equations : $3a + 9c = 153$
                        $5a + 11c = 211$

$$3a + 9c = 153$$
$$5a + 11c = 211$$

ii)   Find the price of one adult and one child ticket.

Enter into GDC :

Let a be x and c be y, then GDC gives

$$x = 18$$
$$y = 11$$

$$a = \$18, \quad c = \$11$$
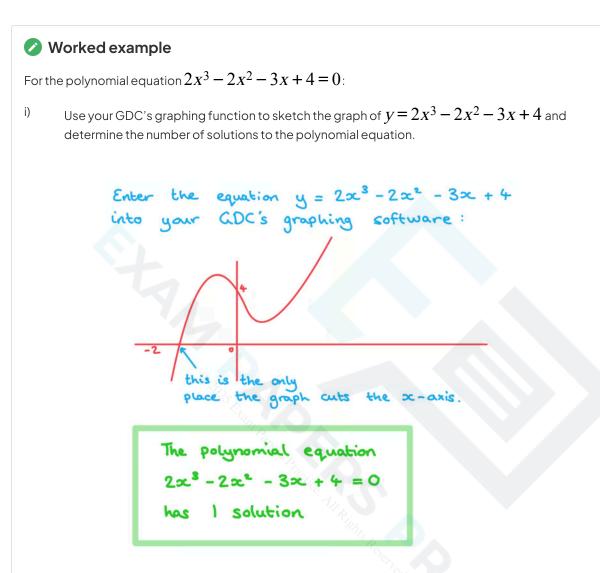
Polynomial Equations

What is a polynomial equation?

- A polynomial is an algebraic expression consisting of a finite number of terms, with non-negative integer indices only
  - It is in the form $ax^n + bx^{n-1} + cx^{n-2} + ..., n \in \mathbb{N}$
- A **polynomial equation** is an equation where a polynomial is equal to zero
- The number of **solutions (roots or zeros)** depend on the **order** of the polynomial equation
  - A polynomial equation of order two can have up to two solutions
  - A polynomial equation of order five can have up to five solutions
- A polynomial equation of an odd degree will always have at least one solution
- A polynomial equation of an even degree could have no solutions

How do I use my GDC to solve polynomial equations?

- You should use your GDC's graphing mode to look at the shape of the polynomial
  - You will be able to see the number of solutions
  - This will be the number of times the graph cuts through or touches the x-axis
  - When entering a function into the graphing section you may need to adjust your zoom settings to be able to see the full graph on your display
  - Whilst in this mode you can then choose to **analyse** the graph
  - This will give you the option to see the solutions of the polynomial equation
    - This may be written as the **zeros** (points where the graph meets the x-axis)
- Your GDC will also have a function within the algebra menu to solve polynomial equations
  - You will need to enter the **order (highest degree)** of the polynomial
  - This is the highest power (or exponent) in the equation
  - Enter the equation into your calculator
  - Your GDC will display the solutions (roots) of the equation
    - Be aware that your GDC may either show all solutions or only the first solution, it is always worth plotting a graph of the function to check how many solutions there should be

✏ Worked example

For the polynomial equation $2x^3 - 2x^2 - 3x + 4 = 0$:

i) Use your GDC's graphing function to sketch the graph of $y = 2x^3 - 2x^2 - 3x + 4$ and determine the number of solutions to the polynomial equation.

Enter the equation $y = 2x^3 - 2x^2 - 3x + 4$ into your GDC's graphing software:

this is the only place the graph cuts the x-axis.

The polynomial equation
$2x^3 - 2x^2 - 3x + 4 = 0$
has 1 solution

ii) Use your GDC to find the solution(s) of the polynomial equation.

Use your GDC's graph analysis tool to find the 'zeros'.

$$x = -1.3101\ldots$$

$$\boxed{x = -1.31 \quad (3sf)}$$

Alternative method:

Enter the equation $2x^3 - 2x^2 - 3x + 4 = 0$ into your GDC's equation solving mode.